REAL ANALYSIS (PREVIOUS PAPERS WITH SOLUTIONS)

GATE – 2006

1. Let E be a non-measurable subset of $[0, 1]$. If $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} \frac{-1}{2}, & x \in E \\ 0, & \text{otherwise} \end{cases}$. Then, **(1 mark)**

(A) f is measurable but not $|f|$
(B) $|f|$ is measurable but not f
(C) both f and $|f|$ are measurable
(D) neither f nor $|f|$ is measurable

2. Let $L^2([0, 1])$ denote the space of all square integrable functions on $[0, 1]$.

Define $f_1, f_2 : [0, 1] \to \mathbb{R}$ by $f_1(t) = \begin{cases} t^{-\frac{1}{3}}, & 0 < t \le 1 \\ 0, & t = 0 \end{cases}$, $f_2(t) = \begin{cases} t^{\frac{2}{3}}, & 0 < t \le 1 \\ 0, & t = 0 \end{cases}$. Then, **(1 mark)**

(A) f_1 belongs to $L^2([0, 1])$ but not f_2
(B) f_2 belongs to $L^2([0, 1])$ but not f_1
(C) both f_1 and f_2 belongs to $L^2([0,1])$
(D) neither f_1 nor f_2 belongs to $L^2([0,1])$

3. Let $S, T \subseteq \mathbb{R}^2$ be given by $S = \left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \le 1\right\} \cup \{(0,0)\}$ and

$T = \left\{\left(x, x\sin\frac{1}{x}\right) : 0 < x \le 1\right\} \cup \{(0,0)\}$. Then under the usual metric on $\mathbb{R}^2$, **(2 marks)**

(A) S is compact but not T
(B) T is compact but not S
(C) both S and T are compact
(D) neither S nor T is compact

4. Let $S, T \subseteq \mathbb{R}$ be given by $S = \left\{x \in \mathbb{R} : 2x^2 \cos\frac{1}{x} = 1\right\}$ and $T = \left\{x \in \mathbb{R} : 2x^2 \cos\frac{1}{x} \le 1\right\} \cup \{0\}$. Then, **(2 marks)**

under the usual metric on $\mathbb{R}$,
(A) S is complete but not T
(B) T is complete but not S
(C) both S and T are complete
(D) neither S nor T is complete

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} n, & \text{if } x = n, n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$ and $T = \mathbb{N} \cup \left\{n + \frac{1}{n} ; n \in \mathbb{N}\right\}$. Then, **(2 marks)**

under the usual metric on $\mathbb{R}$, f is uniformly continuous on
(A) $\mathbb{N}$ but not T
(B) T but not $\mathbb{N}$
(C) both $\mathbb{N}$ and T
(D) neither $\mathbb{N}$ nor T

6. For each $n \in \mathbb{N}$, and $n > 1$, define $f_n [0, 1] \to \mathbb{R}$ by $f_n(x) = \begin{cases} |nx - 1| & \text{for } 0 \le x < \frac{2}{n} \\ 1 & \text{for } \frac{2}{n} \le x < 1 \end{cases}$.

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Let $g_1, g_2 : [0,1] \to \mathbb{R}$ be defined by $g_1(x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$ and $g_2(x) = 1$ for $0 \le x \le 1$. Then, on $[0,1]$

(A) $f_n \to g_1$ pointwise but not uniformly (B) $f_n \to g_2$ pointwise but not uniformly (2 marks)

(C) $f_n \to g_1$ uniformly (D) $f_n \to g_2$ uniformly

7. Let $f_n, g_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^2(1-x^2)^{n-1}$ and $g_n(x) = \dfrac{1}{n^2(1+x^2)}$ for $n \in \mathbb{N}$. Then, on $[0, 1]$, (2 marks)

(A) $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly but not $\sum_{n=1}^{\infty} g_n(x)$

(B) $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly but not $\sum_{n=1}^{\infty} f_n(x)$

(C) both $\sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} g_n(x)$ converge uniformly

(D) neither $\sum_{n=1}^{\infty} f_n(x)$ nor $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly

8. The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \int_0^x (2\sin^4 t \cos^2 t)\, dt$ is (2 marks)

(A) not continuous

(B) continuous but not uniformly

(C) uniformly continuous but not Lipschitz continuous

(D) Lipschitz continuous

9. Let S be a non-measurable subset of $\mathbb{R}$ and T be a measurable subset of $\mathbb{R}$ such that $S \subset T$. Denote the outer measure of a set U by $m^*(U)$. Then, (2 marks)

(A) $m^*(T/S) = 0$ and $m^*(S) = 0$ (B) $m^*(T/S) > 0$ and $m^*(S) > 0$

(C) $m^*(T/S) > 0$ and $m^*(S) = 0$ (D) $m^*(T/S) = 0$ and $m^*(S) > 0$

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$. The directional derivative of f at $(0, 0)$ in the direction of the vector $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ is (2 marks)

(A) $\dfrac{1}{\sqrt{2}}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{2\sqrt{2}}$ (D) $\dfrac{1}{4\sqrt{2}}$

11. Consider the hemisphere $x^2 + y^2 + (z-2)^2 = 9$, $2 \le z \le 5$ and the vector field $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$. The surface integral $\iint (\vec{F} \cdot \vec{n})\, d\sigma$, evaluated over the hemisphere with $\vec{n}$ denoting the unit outward normal, is (2 marks)

(A) 9π (B) 27π (C) 54π (D) 162π

Statement for Linked Answer Questions 12 and 13:

Let $p(x) = c_0 + c_1 x$ minimize $\langle f(x) - p(x), f(x) - p(x) \rangle = \int_{-1}^{1} (f(x) - p(x))^2\, dx$ over all polynomials of degree less than or equal to 1.

12. The best choice of coefficients c_0, c_1 is (2 marks)

(A) $\langle f, 1 \rangle, \langle f, x \rangle$ (B) $\langle f, 1 \rangle, \frac{2}{3}\langle f, x \rangle$

(C) $\frac{1}{2}\langle f, 1 \rangle, \frac{3}{2}\langle f, x \rangle$ (D) $\frac{2}{3}\langle f, 1 \rangle, \frac{2}{3}\langle f, x \rangle$

13. If $f(x) = x^2 + x$, then $p(x)$ is given by (2 marks)

(A) $\left(1 + \dfrac{x}{3}\right)$ (B) $\dfrac{1}{3}(1 + 3x)$ (C) $\dfrac{1}{3}(1 + x)$ (D) $\dfrac{2}{3}(1 + x)$

GATE – 2007

14. For a positive integer n, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \begin{cases} \dfrac{1}{4n+5}, & \text{if } 0 \le x \le n \\ 0, & \text{otherwise} \end{cases}$. (1 mark)

Then $\{f_n(x)\}$ converges to zero

(A) uniformly but not in L^1 norm (B) uniformly and also in L^1 norm

(C) pointwise but not uniformly (D) in L^1 norm but not pointwise

15. Let P_1 and P_2 be two projection operators on a vector space. Then (1 mark)

(A) $P_1 + P_2$ is a projection if $P_1P_2 = P_2P_1 = 0$ (B) $P_1 - P_2$ is a projection if $P_1P_2 = P_2P_1 = 0$

(C) $P_1 + P_2$ is a projection (D) $P_1 - P_2$ is a projection

16. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} 1, & \text{if } xy = 0 \\ 2, & \text{otherwise} \end{cases}$, if $S = \{(x, y) : f \text{ is continuous at the point } (x, y)\}$, (1 mark)

then

(A) S is open (B) S is connected (C) $S = \phi$ (D) S is closed

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17. Let $\{E_n : n = 1, 2, \ldots\}$ be a decreasing sequence of Lebesgue measurable sets on $\mathbb{R}$ and let F be a Lebesgue measurable set on $\mathbb{R}$ such that $E_1 \cap F = \phi$. Suppose that F has Lebesgue measure 2 and the Lebesgue measure of E_n equals $\frac{2n+2}{3n+1}, n = 1, 2, \ldots$. Then the Lebesgue measure of the set $\left(\bigcap_{n=1}^{\infty} E_n\right) \cup F$ equals (1 mark)

(A) $\frac{5}{3}$ (B) 2 (C) $\frac{7}{3}$ (D) $\frac{8}{3}$

8. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin(nx)$. Then the series (2 marks)

(A) Converges uniformly on $\mathbb{R}$
(B) Converges pointwise but NOT uniformly on $\mathbb{R}$
(C) Converges in L^1 norm to an integrable function on $[0, 2\pi]$ but does NOT converge uniformly on $\mathbb{R}$.
(D) does NOT converge pointwise

The value of $\int_0^{\infty} \int_{1/y}^{\infty} x^4 e^{-x^3 y} dx dy$ equals (2 marks)

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1

$\lim_{n \to \infty} \left[(n+1) \int_0^1 x^n \ln(1+x) dx \right] =$ (2 marks)

(A) 0 (B) ln2 (C) ln3 (D) ∞

Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x^4, & \text{if } x \text{ is rational} \\ 2x^2 - 1, & \text{if } x \text{ is irrational} \end{cases}$. Let S be the set of points where f is continuous. Then (2 marks)

(A) $S = \{1\}$ (B) $S = \{-1\}$ (C) $S = \{-1, 1\}$ (D) $S = \phi$

12. For a positive real number p, let $\{f_n : n = 1, 2, \ldots\}$ be sequence of functions defined on $[0,1]$ by $f_n(x) = \begin{cases} n^{p+1} x, & \text{if } 0 \le x \le \frac{1}{n} \\ \frac{1}{x^p}, & \text{if } \frac{1}{n} < x \le 1 \end{cases}$. Let $f(x) = \lim_{n \to \infty} f_n(x), x \in [0,1]$. Then, on $[0, 1]$. (2 marks)

(A) f is Riemann integrable
(B) the improper integral $\int_0^1 f(x) dx$ converges for $p \ge 1$

(C) the improper integral $\int_0^1 f(x) dx$ converges for $p < 1$.
(D) f_n converges uniformly

23. Which of the following inequality is NOT true for $x \in \left(\frac{1}{4}, \frac{3}{4}\right)$? (2 marks)

(A) $e^{-x} > \sum_{j=0}^{2} \frac{(-x)^j}{j!}$ (B) $e^{-x} < \sum_{j=0}^{3} \frac{(-x)^j}{j!}$ (C) $e^{-x} > \sum_{j=0}^{4} \frac{(-x)^j}{j!}$ (D) $e^{-x} > \sum_{j=0}^{5} \frac{(-x)^j}{j!}$

Statement for linked Answer Questions 24 and 25:
Suppose $E = \{(x, y) : xy \ne 0\}$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & \text{if } xy = 0 \\ y \sin\left(\frac{1}{x}\right) + x \sin\left(\frac{1}{y}\right), & \text{otherwise} \end{cases}.$$

Let S_1 be the set of points in $\mathbb{R}^2$, where f_x exists and S_2 be the set of points in $\mathbb{R}^2$, where f_y exists. Also, let E_1 be the set of points where f_x is continuous and E_2 be the set of points where f_y is continuous.

24. S_1 and S_2 are given by (2 marks)
(A) $S_1 = E \cup \{(x, y) : y = 0\}$, $S_2 = E \cup \{(x, y) : x = 0\}$
(B) $S_1 = E \cup \{(x, y) : x = 0\}$, $S_2 = E \cup \{(x, y) : y = 0\}$
(C) $S_1 = S_2 = \mathbb{R}^2$
(D) $S_1 = S_2 = E \cup \{(0,0)\}$

25. E_1 and E_2 are given by (2 marks)
(A) $E_1 = E_2 = S_1 \cap S_2$
(B) $E_1 = E_2 = S_1 \cap S_2 \setminus \{(0, 0)\}$
(C) $E_1 = S_1$, $E_2 = S_2$
(D) $E_1 = S_2$, $E_2 = S_1$

GATE - 2008

26. Let $E = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le x\}$. Then $\iint_E (x + y) dx dy$ is equal to (1 mark)

(A) –1 (B) 0 (C) $\frac{1}{2}$ (D) 1

27. For $(x,y) \in \mathbb{R}^2$, let $f(x,y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & \text{if } (x,y) \ne (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$. Then (1 mark)

(A) f_x and f_y exist at (0,0) and f is continuous at (0,0)

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(B) f_x and f_y exist at (0,0) and f is discontinuous at (0,0)

(C) f_x and f_y do not exist at (0,0) and f is continuous at (0,0)

(D) f_x and f_y do not exist at (0,0) and f is discontinuous at (0,0)

28. Which one of the following does NOT hold for all continuous functions $f : [-\pi,\pi] \to \mathbb{C}$? **(2 marks)**

(A) If $f(-t) = f(t)$ for each $t \in [-\pi,\pi]$, then $\int_{-\pi}^{\pi} f(t)\,dt = 2\int_{0}^{\pi} f(t)\,dt$

(B) If $f(-t) = -f(t)$ for each $t \in [-\pi, \pi]$, then $\int_{-\pi}^{\pi} f(t)\,dt = 0$

(C) If $f(-t) = -f(t)$ for each $t \in [-\pi,\pi]$, then $\int_{-\pi}^{\pi} f(-t)dt = -\int_{-\pi}^{\pi} f(t)dt$

(D) There is an α with $-\pi < \alpha < \pi$ such that $\int_{-\pi}^{\pi} f(t)\,dt = 2\pi f(\alpha)$

29. Which one of the following statement holds? **(2 marks)**

(A) The series $\sum_{n=0}^{\infty} x^n$ converges for each $x \in [-1,1]$

(B) The series $\sum_{n=0}^{\infty} x^n$ converges uniformly for each $x \in [-1,1]$

(C) The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for each $x \in [-1,1]$

(D) The series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly for each $x \in [-1,1]$

30. For $x \in [-\pi,\pi]$, let $f(x) = (\pi+x)(\pi-x)$ and $g(x) = \begin{cases} \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$, consider the statements

P: The Fourier series of f converges uniformly to f on $[-\pi,\pi]$

Q: The Fourier series of g converges uniformly to g on $[-\pi,\pi]$. Then **(2 marks)**

(A) P and Q are true (B) P is true but Q is false

(C) P is false but Q is true (D) both P and Q are false

31. Let $W = \{(x,y,z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4\}$ and $F : W \to \mathbb{R}^3$ be defined by $F(x,y,z) = \frac{(x,y,z)}{[x^2+y^2+z^2]^{3/2}}$ for $(x,y,z) \in W$. If ∂W denotes the boundary of W oriented by the outward normal n to W, then $\iint_{\partial W} F.n\,dS$ is equal to **(2 marks)**

(A) 0 (B) 4π (C) 8π (D) 12π

32. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be a measurable function such that $|f_n(t)| \le \frac{1}{\sqrt{t}}$ for all $t \in (0,1)$. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(t) = 1$ if t is irrational and $f(t) = -1$ if t is rational. Assume that $f_n(t) \to f(t)$ as $n \to \infty$ for all $t \in [0,1]$. Then **(2 marks)**

(A) f is not measurable

(B) f is measurable and $\int_{[0,1]} f_n d\mu \to 1$ as $n \to \infty$

(C) f is measurable and $\int_{[0,1]} f_n d\mu \to 0$ as $n \to \infty$

(D) f is measurable and $\int_{[0,1]} f_n d\mu \to -1$ as $n \to \infty$

33. Let $X = \mathbb{R}$ and let $\Im = \{U \subseteq X : X - U \text{ is finite}\} \cup \{\phi, X\}$. The sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ in $(X, \Im)$ **(2 marks)**

(A) converges to 0 and not to any other point of X

(B) does not converge to 0

(C) converges to each point of X

(D) is not convergent in X

34. Let $E = \{(x,y) \in \mathbb{R}^2 : |x| \le 1, |y| \le 1\}$. Define $f : E \to \mathbb{R}$ by $f(x,y) = \frac{x+y}{1+x^2+y^2}$. Then the range of f is a **(2 marks)**

(A) connected open set (B) connected closed set

(C) bounded open set (D) closed and unbounded set

35. Which one of the following subsets of $\mathbb{R}$ (with the usual metric) is NOT complete? **(2 marks)**

(A) $[1, 2] \cup [3,4]$ (B) $[0,\infty)$ (C) $[0,1)$ (D) $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

GATE – 2009

36. A function $f : \mathbb{R} \to \mathbb{R}$ need not be Lebesgue measurable if **(1 mark)**

(A) f is monotone

(B) $\{x \in R : f(x) \ge \alpha\}$ is measurable for each $\alpha \in \mathbb{Q}$.

(C) $\{x \in R : f(x) \ge \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$

(D) For each open set G in $\mathbb{R}$, $f^{-1}(G)$ is measurable.

37. Which of the following sequence $\{f_n\}_{n=1}^{\infty}$ of functions does not converges uniformly on $[0, 1]$? **(2 marks)**

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(A) $f_n(x)=\frac{e^{-x}}{n}$ (B) $f_n(x)=(1-x)^n$

(C) $f_n(x)=\frac{x^2+nx}{n}$ (D) $f_n(x)=\frac{\sin(nx+n)}{n}$

38. Let $E=\{(x,y)\in\mathbb{R}^2 : 0<x<y\}$. Then $\iint_E ye^{-(x+y)}dxdy=$ **(2 marks)**

(A) $\frac{1}{4}$ (B) $\frac{3}{2}$ (C) $\frac{4}{3}$ (D) $\frac{3}{4}$

39. Let $f_n(x)=\frac{1}{n}\sum_{k=0}^{n}\sqrt{k(n-k)}\binom{n}{k}x^k(1-x)^{n-k}$ for $x\in[0,1], n=1,2,\ldots$. If $\lim_{n\to\infty} f_n(x)=f(x)$ for $x\in[0,1]$, then the maximum value of $f(x)$ on $[0,1]$ is **(2 marks)**

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Statement for linked Answer Questions 40 and 41:

Let $f(x, y) = kxy-x^3y-xy^3$ for $(x, y)\in\mathbb{R}^2$, where k is a real constant. The directional derivative of f at the point (1, 2) in the direction of the unit vector $u=\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ is $\frac{15}{\sqrt{2}}$.

40. The value of k is **(2 marks)**

(A) 2 (B) 4 (C) 1 (D) -2

41. The value of f at local minimum in the rectangular region $R=\left\{(x,y)\in\mathbb{R}^2 : |x|<\frac{3}{2}, |y|<\frac{3}{2}\right\}$ is **(2 marks)**

(A) -2 (B) -3 (C) $\frac{-7}{8}$ (D) 0

GATE - 2010

42. Which one of the following sets of functions is not orthogonal (with respect to the L^2 – inner product) over the given interval? **(1 mark)**

(A) $\{\sin nx : n\in\mathbb{N}\}, -\pi<x<\pi$ (B) $\{\cos nx : n\in\mathbb{N}\}, -\pi<x<\pi$

(C) $\{x^{\frac{2n+1}{2}} : n\in\mathbb{N}\}, -1<x<1$ (D) $\{x^{2n+1} : n\in\mathbb{N}\}, -1<x<1$

43. If $f:[1,2]\to\mathbb{R}$ is a non negative Riemann integrable function such that $\int_1^2 \frac{f(x)}{\sqrt{x}}dx = k\int_1^2 f(x)dx \neq 0$ then k belongs to the interval **(1 mark)**

(A) $\left[0,\frac{1}{3}\right]$ (B) $\left(\frac{1}{3},\frac{2}{3}\right]$ (C) $\left(\frac{2}{3},1\right]$ (D) $\left(1,\frac{4}{3}\right]$

44. The set $X=\mathbb{R}$ with the metric $d(x,y)=\frac{|x-y|}{1+|x-y|}$ is **(1 mark)**

(A) bounded but not compact (B) bounded but not complete

(C) complete but not bounded (D) compact but not complete

45. Let $f(x,y)=\begin{cases}\frac{xy}{(x^2+y^2)^{5/2}}\left[1-\cos(x^2+y^2)\right], & (x,y)\neq(0,0)\\ k, & (x,y)=(0,0)\end{cases}$

Then the value of k for which $f(x,y)$ is continuous at (0,0) is **(1 mark)**

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$

46. Let A and B be disjoint subsets of $\mathbb{R}$ and let m^* denote the Lebesgue outer measure on $\mathbb{R}$. Consider the statements:

P: $m^*(A\cup B)=m^*(A)+m^*(B)$

Q: Both A and B are Lebesgue measurable

R: One of A and B is Lebesgue measurable. Which one of the following is correct? **(1 mark)**

(A) If P is true, then Q is true (B) If P is NOT true, then R is true

(C) If R is true, then P is NOT true (D) If R is true, then P is true

47. Let $f:\mathbb{R}\to[0,\infty)$ be a Lebesgue measurable function and E be a Lebesgue measurable subset of $\mathbb{R}$ such that $\int_E f\,dm=0$, where m is the Lebesgue measure on $\mathbb{R}$. Then **(1 mark)**

(A) $m(E)=0$ (B) $\{x\in\mathbb{R}: f(x)=0\}=E$

(C) $m(\{x\in E : f(x)\neq 0\})=0$ (D) $m(\{x\in E : f(x)=0\})=0$

48. Let $\{f_n\}$ be a sequence of real valued differentiable functions on [a,b] such that $f_n(x)\to f(x)$ as $n\to\infty$ for every $x\in[a,b]$ and for some Riemann–integrable function $f:[a,b]\to\mathbb{R}$. Consider the statements

P_1 : $\{f_n\}$ converges uniformly

P_2 : $\{f_n'\}$ converges uniformly

P_3 : $\int_a^b f_n(x)dx\to\int_a^b f(x)dx$

P_4 : f is differentiable

Then which one of the following need NOT be true ? **(2 marks)**

(A) P_1 implies P_3 (B) P_2 implies P_1 (C) P_2 implies P_4 (D) P_3 implies P_1

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49. Let $f_n(x)=\frac{x^n}{1+x}$ and $g_n(x)=\frac{x^n}{1+nx}$ for $x\in[0,1]$ and $n\in\mathbb{N}$. Then on the interval $[0,1]$ (2 marks)

(A) both $\{f_n\}$ and $\{g_n\}$ converges uniformly
(B) neither $\{f_n\}$ nor $\{g_n\}$ converges uniformly
(C) $\{f_n\}$ converges uniformly but $\{g_n\}$ does not converge uniformly
(D) $\{g_n\}$ converges uniformly but $\{f_n\}$ does not converge uniformly

50. Consider the power series $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{x^n}{n}$. Then (2 marks)

(A) Both converge on $(-1,1]$ (B) Both converge on $[-1,1)$
(C) Exactly one of them converges on $(-1,1]$ (D) None of them converges on $[-1,1)$

51. Let $I=\oint_C \frac{e^y}{x}dx+(e^y\ell nx+x)dy$, where C is the positively oriented boundary of the region enclosed by $y=1+x^2, y=2, x=\frac{1}{2}$. Then the value of I is (2 marks)

(A) $\frac{1}{8}$ (B) $\frac{5}{24}$ (C) $\frac{7}{24}$ (D) $\frac{3}{8}$

GATE - 2011

52. The series $\sum_{n=1}^{\infty}x^{\ln n}, x>0$, is convergent on the interval (1 mark)

(A) (0,1/e) (B) (1/e,e) (C) (0,e) (D) (1,e)

53. For $0\le x\le 1$, let $f_n(x)=\begin{cases}\frac{n}{1+n}, & \text{if } x \text{ is irrational}\\ 0, & \text{if } x \text{ is rational}\end{cases}$ and $f(x)=\lim_{n\to\infty}f_n(x)$.
Then, on the interval $[0,1]$ (2 marks)

(A) f is measurable and Riemann integrable (B) f is measurable and Lebesgue integrable
(C) f is not measurable (D) f is not Lebesgue integrable

54. If x, y and z are positive real numbers, then the minimum value of $x^2+8y^2+27z^2$, where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ is (2 marks)

(A) 108 (B) 216 (C) 405 (D) 1048

Statement for linked Answer Questions 55 and 56:

Let $f_n(x)=\frac{x}{\{(n-1)x+1\}\{nx+1\}}$ and $s_n(x)=\sum_{j=1}^{n}f_j(x)$ for $x\in[0,1]$.

55. The sequence $\{s_n\}$ (2 marks)

(A) converges uniformly on $[0,1]$
(B) converges pointwise on $[0,1]$ but not uniformly
(C) converges pointwise for $x=0$ but not $x\in(0,1]$
(D) does not converge for $x\in[0,1]$

56. $\lim_{n\to\infty}\int_0^1 s_n(x)dx=1$ (2 marks)

(A) by dominated convergence theorem
(B) by Fatou's lemma
(C) by the fact that $\{s_n\}$ converges uniformly on $[0,1]$
(D) by the fact that $\{s_n\}$ converges pointwise on $[0,1]$

GATE - 2012

57. The Lebesgue measure of the set $A=\left\{0<x\le 1: x\sin\left\{\frac{\pi}{2x}\right\}\ge 0\right\}$ is (2 marks)

(A) 0 (B) 1 (C) $\ln 2$ (D) $1-\ln\sqrt{2}$

58. Which of the following statements are TRUE?

P: The set $\{x\in\mathbb{R}:|\cos x|\le\frac{1}{2}\}$ is complete.
Q: The set $\{x\in\mathbb{R}:\tan x$ is not differentiable$\}$ is complete
R: The set $\{x\in\mathbb{R}:\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ is convergent$\}$ is bounded.
S: The set $\{x\in\mathbb{R}:f(x)=\cos x$ has a local maxima$\}$ is closed. (2 marks)

(A) P and Q (B) R and S (C) Q and S (D) P and S

59. The maximum value of the function $f(x,y,z)=xyz$ subject to the constraint $xy+yz+zx-a=0$, $a>0$ is (2 marks)

(A) $a^{\frac{3}{2}}$ (B) $(a/3)^{3/2}$ (C) $(3/a)^{3/2}$ (D) $(3a/2)^{3/2}$

60. The flux of the vector field $\vec{u}=x\hat{i}+y\hat{j}+z\hat{k}$ flowing out through the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, a>b>c>0$, is (2 marks)

(A) πabc (B) $2\pi abc$ (C) $3\pi abc$ (D) $4\pi abc$

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GATE - 2013

61. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with $f(1)=5$ and $f(3)=11$. If $g(x)=\int_1^3 f(x+t)dt$, then $g'(0)$ is equal to _____ **(1 mark)**

62. Let $f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}$. Then **(1 mark)**

(A) $\lim_{x\to 0} f(x)=0$ (B) $\lim_{x\to 0} f(x)=1$

(C) $\lim_{x\to 0} f(x)=\frac{\pi^2}{6}$ (D) $\lim_{x\to 0} f(x)$ does not exist

63. Let $x_0=0$. Define $x_{n+1}=\cos x_n$ for every $n\geq 0$. Then **(1 mark)**

(A) $\{x_n\}$ is increasing and convergent

(B) $\{x_n\}$ is decreasing and convergent

(C) $\{x_n\}$ is convergent and $x_{2n}<\lim_{m\to\infty} x_m<x_{2n+1}$ for every $n\in\mathbb{N}$

(D) $\{x_n\}$ is not convergent

64. Let $\{a_n\}$ be the sequence of consecutive positive solutions of the equation $\tan x=x$ and let $\{b_n\}$ be the sequence of consecutive positive solutions of the equation $\tan\sqrt{x}=x$. Then **(1 mark)**

(A) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ converges but $\sum_{n=1}^{\infty}\frac{1}{b_n}$ diverges (B) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ diverges but $\sum_{n=1}^{\infty}\frac{1}{b_n}$ converges

(C) Both $\sum_{n=1}^{\infty}\frac{1}{a_n}$ and $\sum_{n=1}^{\infty}\frac{1}{b_n}$ converges (D) Both $\sum_{n=1}^{\infty}\frac{1}{a_n}$ and $\sum_{n=1}^{\infty}\frac{1}{b_n}$ diverges

65. The value of the integral $\int_0^{\infty}\int_x^{\infty}\left(\frac{1}{y}\right)e^{-y/2}dydx$ is _____ **(1 mark)**

66. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $f(x,y)=(e^{x+y}, e^{x-y})$. The area of the image of the region $\{(x,y)\in\mathbb{R}^2: 0<x,y<1\}$ under the mapping f is **(1 mark)**

(A) 1 (B) e-1 (C) e^2 (D) e^2-1

67. Let $f,g:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}x & \text{if } x=\frac{1}{n} \text{ for } n\in\mathbb{N}\\ 0 & \text{otherwise}\end{cases}$ and $g(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{otherwise}\end{cases}$. Then **(2 marks)**

(A) Both f and g are Riemann integrable

(B) f is Riemann integrable and g is Lebesgue integrable

(C) g is Riemann and f is Lebesgue integrable

(D) Neither f nor g is Riemann integrable

68. Which of the following subsets of $\mathbb{R}^2$ is NOT compact? **(2 marks)**

(A) $\{(x,y)\in\mathbb{R}^2: -1\leq x\leq 1, y=\sin x\}$

(B) $\{(x,y)\in\mathbb{R}^2: -1\leq x\leq 1, y=x^8-x^3-1\}$

(C) $\{(x,y)\in\mathbb{R}^2: y=0, \sin(e^{-x})=0\}$

(D) $\{(x,y)\in\mathbb{R}^2: x>0,\ y=\sin\left(\frac{1}{x}\right)\}\cap\{(x,y)\in\mathbb{R}^2: x>0, y=\frac{1}{x}\}$

69. The value of the limit $\lim_{n\to\infty}\frac{2^{-n^2}}{\sum_{k=n+1}^{\infty}2^{-k^2}}$ is **(2 marks)**

(A) 0 (B) some $c\in(0,1)$ (C) 1 (D) ∞

70. Let $S=\{x\in\mathbb{R}: x\geq 0, \sum_{n=1}^{\infty}x^{\sqrt{n}}<\infty\}$. Then the supremum of S is **(2 marks)**

(A) 1 (B) 1/e (C) 0 (D) ∞

71. Which of the following statements are TRUE?

P: If $f\in L^1(\mathbb{R})$, then f is continuous

Q: If $f\in L^1(\mathbb{R})$ and $\lim_{|x|\to\infty} f(x)$ exists, then the limit is zero

R: If $f\in L^1(\mathbb{R})$, then f is bounded.

S: If $f\in L^1(\mathbb{R})$ is uniformly continuous, then $\lim_{|x|\to\infty} f(x)$ exists and equals zero. **(2 marks)**

(A) Q and S only (B) P and R only (C) P and Q only (D) R and S only

Statement for Linked Answer Questions 72 and 73:

Let $X=\{(x,y)\in\mathbb{R}^2: x^2+y^2=1\}\cup([-1,1]\times\{0\})\cup(\{0\}\times[-1,1])$. Let $n_0=\max\{k: k<\infty,$ there are k distinct points $p_1,\ldots,p_k\in X$ such that $X\setminus\{p_1,\ldots,p_k\}$ is connected$\}$

72. The value of n_0 is _____ **(2 marks)**

73. Let $q_1,\ldots,q_{n_0+1}$ be n_0+1 distinct points and $Y=X\setminus\{q_1,\ldots,q_{n_0+1}\}$. Let m be the number of connected components of Y. The maximum possible value of m is _____ **(2 marks)**

GATE - 2014

74. For the sequence of functions $f_n(x)=\frac{1}{x^2}\sin\left(\frac{1}{nx}\right)$, $x\in[1,\infty)$, consider the following quantities expressed in terms of Lebesgue integrals **(1 mark)**

I: $\lim_{n\to\infty}\int_1^{\infty} f_n(x)dx$, II: $\int_1^{\infty}\lim_{n\to\infty} f_n(x)dx$. Which of the following is TRUE?

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(A) The limit in I does not exist
(B) The integrand in II is not integrable on $[1,\infty)$
(C) Quantities I and II are well-defined, but $I \neq II$
(D) Quantities I and II are well-defined and $I = II$

75. Let d_1, d_2 and d_3 be metrics on a set X with at least two elements. Which of the following is NOT a metric on X? **(1 mark)**

(A) $\min\{d_1, 2\}$ (B) $\max\{d_1, 2\}$ (C) $\frac{d_3}{1+d_3}$ (D) $\frac{d_1+d_2+d_3}{3}$

76. Let $f(x) = \begin{cases} -3\pi, & \text{if } -\pi < x \le 0 \\ 3\pi, & \text{if } 0 < x < \pi \end{cases}$ be a periodic function of period 2π. The coefficient of $\sin 3x$ in the Fourier series expansion of $f(x)$ on the interval $[-\pi, \pi]$ is _____ **(1 mark)**

77. Let $\vec{F}$ be a vector field defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by $\vec{F}(x,y) = \frac{y}{x^2+y^2}\hat{i} - \frac{x}{x^2+y^2}\hat{j}$.

Let $\gamma, \alpha : [0,1] \to \mathbb{R}^2$ be defined by $\gamma(t) = (8\cos 2\pi t, 17\sin 2\pi t)$ and $\alpha(t) = (26\cos 2\pi t, -10\sin 2\pi t)$. If $3\int_{\gamma} \vec{F}\cdot d\vec{r} - 4\int_{\alpha} \vec{F}\cdot d\vec{r} = 2m\pi$, then m is _____ **(2 marks)**

78. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $g(x,y,z) = (3y+4z, 2x-3z, x+3y)$ and let $S = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. If $\iiint_{g(S)} (2x+y-2z)\,dx\,dy\,dz = \alpha \iiint_S z\,dx\,dy\,dz$, then α is _____ **(2 marks)**

79. The value of $\iint_R xy\,dxdy$, where R is the region in the first quadrant bounded by the curves $y = x^2$, $y + x = 2$ and $x = 0$ is _____ **(2 marks)**

GATE - 2015

80. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = \int_0^x \sin^2(t^2)dt$. Then the function f is **(1 mark)**

(A) Uniformly continuous on $[0,1)$ but NOT on $(0,\infty)$
(B) Uniformly continuous on $(0,\infty)$ but NOT on $[0,1)$
(C) Uniformly continuous on both $[0,1)$ and $(0,\infty)$
(D) Neither uniformly continuous on $[0,1)$ nor uniformly continuous on $(0,\infty)$

81. Let d_1 and d_2 denote the usual metric and the discrete metric on $\mathbb{R}$, respectively. Let $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$ be defined by $f(x) = x$, $x \in \mathbb{R}$. Then **(1 mark)**

(A) f is continuous but f^{-1} is NOT continuous
(B) f^{-1} is continuous but f is NOT continuous
(C) Both f and f^{-1} are continuous
(D) Neither f nor f^{-1} is continuous

82. Let $S = \left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \le 1\right\}$ and $T = S \cup \{(0,0)\}$. Under the usual metric on $\mathbb{R}^2$. **(2 marks)**

(A) S is closed but T is NOT closed (B) T is closed but S is NOT closed
(C) both S and T are closed (D) neither S nor T is closed

83. Define $f_1, f_2 : [0,1] \to \mathbb{R}$ by $f_1(x) = \sum_{n=1}^{\infty} \frac{x\sin(n^2x)}{n^2}$ and $f_2(x) = \sum_{n=1}^{\infty} x^n(1-x^n)^{-1}$. Then **(2 marks)**

(A) f_1 is continuous but f_2 is NOT continuous
(B) f_2 is continuous but f_1 is NOT continuous
(C) both f_1 and f_2 are continuous
(D) neither f_1 nor f_2 is continuous

84. Consider the unit sphere $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the unit normal vector $\hat{n} = (x,y,z)$ at each point (x,y,z) on S. The value of the surface integral $\iint_S \left\{\left(\frac{2x}{\pi} + \sin(y^2)\right)x + \left(e^z - \frac{y}{\pi}\right)y + \left(\frac{2z}{\pi} + \sin^2 y\right)z\right\} d\sigma$ is equal to _____. **(2 marks)**

85. Let $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 1000, 1 \le y \le 1000\}$. Define $f(x,y) = \frac{xy}{2} + \frac{500}{x} + \frac{500}{y}$. Then the minimum value of f on D is equal to _____. **(2 marks)**

GATE - 2016

86. Let $S = [0,1) \cup [2,3]$ and $f : S \to \mathbb{R}$ be a strictly increasing function such that $f(S)$ is connected. Which of the following statements is TRUE? **(1 mark)**

(A) f has exactly one discontinuity (B) f has exactly two discontinuities
(C) f has infinitely many discontinuities (D) f is continuous

87. Let $a_1 = 1$ and $a_n = a_{n-1} + 4$, $n \ge 2$. Then, $\lim_{n\to\infty}\left[\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{n-1}a_n}\right]$ is equal to _____ **(1 mark)**

88. Maximum $\{x+y : (x,y) \in \overline{B(0,1)}\}$ is equal to _____ **(1 mark)**

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89. Consider $(\mathbb{R}^2, \|\cdot\|_\infty)$, where $\|(x,y)\|_\infty$ = maximum $\{|x|, |y|\}$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \frac{x+y}{2}$ and $\tilde{f}$ the norm preserving linear extension of f to $(\mathbb{R}^3, \|\cdot\|_\infty)$. Then, $\tilde{f}_{(1,1,1)}$ is equal to_____ (1 mark)

90. $f: [0, 1] \to [0,1]$ is called a shrinking map if $|f(x) - f(y)| < |x - y|$ for all $x, y \in [0,1]$ and a contraction if there exists an $\alpha < 1$ such that $|f(x) - f(y)| \leq \alpha |x - y|$ for all $x, y \in [0,1]$.

Which of the following statements is TRUE for the function $f(x) = x - \frac{x^2}{2}$? (1 mark)

(A) f is both a shrinking map and a contraction
(B) f is shrinking map but NOT a contraction
(C) f is NOT a shrinking map but a contraction
(D) f is Neither a shrinking map nor a contraction

91. Let $S(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ be the Fourier series of the 2π periodic function defined by $f(x) = x^2 + 4\sin(x)\cos(x)$, $-\pi < x \leq \pi$. Then $\left|\sum_{n=0}^{\infty} a_n - \sum_{n=1}^{\infty} b_n\right|$ is equal to_____ (2 marks)

92. Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ and $I_n = \int_1^n \frac{x - [x]}{x^2} dx$. Then, $S_{10} + I_{10}$ is equal to (2 marks)

(A) $\ell n\ 10 + 1$ (B) $\ell n\ 10 - 1$ (C) $\ell n\ 10 - \frac{1}{10}$ (D) $\ell n\ 10 + \frac{1}{10}$

93. For any $(x,y) \in \mathbb{R}^2 \setminus \overline{B(0,1)}$, let $f(x,y) = \text{distance}\left((x,y), \overline{B(0,1)}\right)$ $= \text{infimum}\left\{\sqrt{(x-x_1)^2 + (y-y_1)^2} : (x_1, y_1) \in \overline{B(0,1)}\right\}$. Then, $\| \nabla f(3,4) \|$ is equal to_____ (2 marks)

94. Let $f(x) = \left(\int_0^x e^{-t^2} dt\right)^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$. Then $f'(\sqrt{\pi}) + g'(\sqrt{\pi})$ is equal to_____ (2 marks)

GATE - 2017

95. Let $f_n: [0,1] \to \mathbb{R}$ be given by $f_n(x) = \frac{2x^2}{x^2 + (1-2nx)^2}$, $n = 1, 2, \ldots$. Then the sequence (f_n) (1 mark)

(A) converges uniformly on $[0, 1]$
(B) does NOT converge uniformly on $[0, 1]$ but has a subsequence that converges uniformly on $[0, 1]$
(C) does NOT converge pointwise on $[0, 1]$
(D) converges pointwise on $[0, 1]$ but does NOT have a subsequence that converges uniformly on $[0, 1]$

GATE (2006-2019) REAL ANALYSIS

96. Let $C: x^2 + y^2 = 9$ be the circle in $\mathbb{R}^2$ oriented positively. Then $\frac{1}{\pi}\oint_C \left(3y - e^{\cos x^2}\right) dx + \left(7x + \sqrt{y^4 + 11}\right) dy$ equals_____ (1 mark)

97. Consider the following statements: (1 mark)

(P): There exists an unbounded subset of $\mathbb{R}$ whose Lebesgue measure is equal to 5.
(Q): If $f: \mathbb{R} \to \mathbb{R}$ is continuous and $g: \mathbb{R} \to \mathbb{R}$ is such that $f = g$ almost everywhere on $\mathbb{R}$, then g must be continuous almost everywhere on $\mathbb{R}$.

Which of the above statements hold TRUE?
(A) Both P and Q (B) Only P (C) Only Q (D) Neither P nor Q

98. For $n \in \mathbb{Z}$, define $c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(n-1)x} dx$, where $i^2 = -1$. Then $\sum_{n \in \mathbb{Z}} |c_n|^2$ equals (1 mark)

(A) $\cosh(\pi)$ (B) $\sinh(\pi)$ (C) $\cosh(2\pi)$ (D) $\sinh(2\pi)$

99. Let (x_n) and (y_n) be two sequences in a complete metric space (X, d) such that $d(x_n, x_{n+1}) \leq \frac{1}{n^2}$ and $d(y_n, y_{n+1}) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then (2 marks)

(A) both (x_n) and (y_n) converge
(B) (x_n) converges but (y_n) need NOT converge
(C) (y_n) converges but (x_n) need NOT converge
(D) neither (x_n) nor (y_n) converges

100. Let $f: [0, 1] \to \mathbb{R}$ be given by $f(x) = 0$ if x is rational, and if x is irrational then $f(x) = 9^n$, where n is the number of zeroes immediately after the decimal point in the decimal representation of x. Then the Lebesgue integral $\int_0^1 f(x)\,dx$ equals_____ (2 marks)

101. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \sin\left(\frac{y^2}{x}\right)\sqrt{x^2 + y^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then, at $(0, 0)$, (2 marks)

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(A) f is continuous and the directional derivative of f does NOT exist in some direction
(B) f is NOT continuous and the directional derivatives of f exist in all directions
(C) f is NOT differentiable and the directional derivatives of f exist in all directions
(D) f is differentiable

102. Let D be the region in $\mathbb{R}^2$ bounded by the parabola $y^2 = 2x$ and the line $y = x$. Then $\iint_D 3xy\,dx\,dy$ equals_____ **(2 marks)**

GATE - 2018

103. Let X and Y be metric spaces and let $f : X \to Y$ be a continuous map. For any subset S of X, which one of the following statements is true? **(1 mark)**
(A) If S is open, then $f(S)$ is open
(B) If S is connected, then $f(S)$ is connected
(C) If S is closed, then $f(S)$ is closed
(D) If S is bounded, then $f(S)$ is bounded

104. Let $I = [2, 3)$, J be the set of all rational numbers in the interval $[4, 6]$, K be the Cantor (ternary) set and let $L = \{7 + x : x \in K\}$. Then the Lebesgue measure of the set $I \cup J \cup L$ equals _____. **(1 mark)**

105. Let $u(x,y,z) = x^2 - 2y + 4z^2$ for $(x,y,z) \in \mathbb{R}^3$. Then the directional derivative of u in the direction $\frac{3}{5}\hat{i} - \frac{4}{5}\hat{k}$ at the point $(5, 1, 0)$ is _____. **(1 mark)**

106. Let $D \subset R^2$ denote the closed disc with center at the origin and radius 2. Then $\iint_D e^{-(x^2+y^2)}dxdy =$ **(2 marks)**
(A) $\pi(1 - e^{-4})$ (B) $\frac{\pi}{2}(1 - e^{-4})$ (C) $\pi(1 - e^{-2})$ (D) $\frac{\pi}{2}(1 - e^{-2})$

107. Let S be the surface of the solid $V = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$. Let $\hat{n}$ denote the unit outward normal to S and let $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}, (x, y, z) \in V$. Then the surface integral $\iint_S \vec{F} \cdot \hat{n} dS$ equals _____. **(2 marks)**

108. If $\sin(\pi x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ for $0 < x < 1$, then $(a_0 + a_1)\pi =$ _____. **(2 marks)**

109. For $n = 1, 2, \ldots$, let $f_n(x) = \frac{2nx^{n-1}}{1+x}, x \in [0,1]$. Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx =$ _____. **(2 marks)**

GATE - 2019

110. If $u_n = \int_1^n e^{-t^2}dt, n = 1,2,3,\ldots$, then which one of the following statements is TRUE? **(1 mark)**
(A) Both the sequence $\{u_n\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty} u_n$ are convergent
(B) Both the sequence $\{u_n\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty} u_n$ are divergent
(C) The sequence $\{u_n\}_{n=1}^{\infty}$ is convergent but the series $\sum_{n=1}^{\infty} u_n$ is divergent
(D) $\lim_{n\to\infty} u_n = \frac{2}{e}$

111. Let $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : -1 < x < 1, -1 < y < 1, -1 < z < 1\}$ and $\phi : \Gamma \to \mathbb{R}$ be a function whose all second order partial derivatives exist and are continuous. If ϕ satisfies the Laplace equation $\nabla^2\phi = 0$ for all $(x, y, z) \in \Gamma$, then which one of the following statements is TRUE in Γ? **(1 mark)**
($\mathbb{R}$ is the set of all real numbers, and $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$)
(A) $\vec{\nabla}\phi$ is solenoidal but not irrotational
(B) $\vec{\nabla}\phi$ is irrotational but not solenoidal
(C) $\vec{\nabla}\phi$ is both solenoidal and irrotational
(D) $\vec{\nabla}\phi$ is neither solenoidal nor irrotational

112. Let $X = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}$ and only finitely many x_i's are non-zero$\}$ and $d : X \times X \to \mathbb{R}$ be a metric on X defined by
$d(x, y) = \sup_{i\in\mathbb{N}} |x_i - y_i|$ for $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)$ in X.
($\mathbb{R}$ is the set of all real numbers and $\mathbb{N}$ is the set of all natural numbers)
Consider the following statements:
P : (X, d) is a complete metric space.
Q : The set $\{x \in X : d(\underline{0}, x) \le 1\}$ is compact, where $\underline{0}$ is the zero element of X.
Which of the above statements is/are TRUE? **(1 mark)**
(A) Both P and Q (B) P only (C) Q only (D) Neither P nor Q

113. Consider the following statements:
I. The set $\mathbb{Q} \times \mathbb{Z}$ is uncountable.
II. The set $\{f : f$ is a function from $\mathbb{N}$ to $\{0, 1\}\}$ is uncountable.
III. The set $\{\sqrt{p} : p$ is a prime number$\}$ is uncountable.
IV. For any infinite set, there exists a bijection from the set to one of its proper subsets.
($\mathbb{Q}$ is the set of all rational numbers, $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all natural numbers)
Which of the above statements are TRUE? **(1 mark)**
(A) I and IV only (B) II and IV only (C) II and III only (D) I, II and IV only

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114. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^6 - 2x^2y - x^4y + 2y^2$.
($\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$)
Which one of the following statements is TRUE? **(1 mark)**
(A) f has a local maximum at origin (B) f has a local minimum at origin
(C) f has a saddle point at origin (D) The origin is not a critical point of f

115. Let $\{a_n\}_{n=0}^{\infty}$ be any sequence of real numbers such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is r, then which one of the following statements is necessarily TRUE? **(1 mark)**
(A) $r \geq 1$ or r is infinite (B) $r < 1$
(C) $r = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$ (D) $r = \sum_{n=0}^{\infty} |a_n|^2$

116. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by $g(x, y) = (e^x \cos y, e^x \sin y)$ and $(a,b) = g\left(1, \frac{\pi}{3}\right)$. ($\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$)
Which one of the following statements is TRUE? **(1 mark)**
(A) g is injective
(B) If h is the continuous inverse of g, defined in some neighbourhood of $(a, b) \in \mathbb{R}^2$, such that $h(a,b) = \left(1, \frac{\pi}{3}\right)$, then the Jacobian of h at (a, b) is e^2
(C) If h is the continuous inverse of g, defined in some neighbourhood of $(a, b) \in \mathbb{R}^2$, such that $h(a,b) = \left(1, \frac{\pi}{3}\right)$, then the Jacobian of h at (a, b) is e^{-2}
(D) g is surjective

117. Let $u_n = \frac{n!}{1.3.5...(2n-1)}$, $n \in \mathbb{N}$ (the set of all natural numbers). Then $\lim_{n \to \infty} u_n$ is equal to _____. **(1 mark)**

118. Let f be any polynomial function of degree at most 2 over $\mathbb{R}$ (the set of all real numbers). If the constants a and b are such that $\frac{df}{dx} = a f(x) + 2 f(x+1) + b f(x+2)$, for all $x \in \mathbb{R}$, then $4a + 3b$ is equal to _____ (round off to 2 places of decimal). **(1 mark)**

119. Let L denote the value of the line integral $\oint_C (3x - 4x^2y)dx + (4xy^2 + 2y)dy$, where C, a circle of radius 2 with centre at origin of the xy-plane, is traversed once in the anti-clockwise direction. Then $\frac{L}{\pi}$ is equal to _____. **(1 mark)**

120. The temperature $T : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ at any point $P(x, y, z)$ is inversely proportional to the square of the distance of P from the origin. If the value of the temperature T at the point $R(0,0,1)$ is $\sqrt{3}$, then the rate of change of T at the point $Q(1,1,2)$ in the direction of $\overrightarrow{QR}$ is equal to _____ (round off to 2 places of decimal).
($\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ and $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ denotes $\mathbb{R}^3$ excluding the origin) **(1 mark)**

121. Let f be a continuous function defined on $[0, 2]$ such that $f(x) \geq 0$ for all $x \in [0, 2]$. If the area bounded by $y = f(x)$, $x = 0$, $y = 0$ and $x = b$ is $\sqrt{3+b^2} - \sqrt{3}$, where $b \in (0, 2]$, then $f(1)$ is equal to _____ (round off to 1 place of decimal). **(1 mark)**

122. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.
Consider the following statements:
I. The partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at $(0, 0)$ but are unbounded in any neighbourhood of $(0, 0)$.
II. f is continuous but not differentiable at $(0, 0)$.
III. f is not continuous at $(0, 0)$.
IV. f is differentiable at $(0, 0)$.
($\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$)
Which of the above statements is/are TRUE? **(2 marks)**
(A) I and II only (B) I and IV only (C) IV only (D) III only

123. Consider the sequence $\{g_n\}_{n=1}^{\infty}$ of functions, where $g_n(x) = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $g_n'(x)$ is the derivative of $g_n(x)$ with respect to x.
($\mathbb{R}$ is the set of all real numbers, $\mathbb{N}$ is the set of all natural numbers).
Then which one of the following statements is TRUE? **(2 marks)**
(A) $\{g_n\}_{n=1}^{\infty}$ does **NOT** converge uniformly on $\mathbb{R}$
(B) $\{g_n'\}_{n=1}^{\infty}$ converges uniformly on any closed interval which does **NOT** contain 1
(C) $\{g_n'\}_{n=1}^{\infty}$ converges point-wise to a continuous function on $\mathbb{R}$
(D) $\{g_n'\}_{n=1}^{\infty}$ converges uniformly on any closed interval which does **NOT** contain 0

124. Let R be the region in the xy-plane bounded by the curves $y = x^2$, $y = 4x^2$, $xy = 1$ and $xy = 5$.
Then the value of the integral $\iint_R \frac{y^2}{x} dydx$ is equal to _____. **(2 marks)**

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REAL ANALYSIS

(SOLUTIONS)

1. **Solution: (D)**

As $f^{-1}\left(-\frac{1}{2}\right)$ and $|f|^{-1}\left(\frac{1}{2}\right)$ are not measurable.

2. **Solution: (A)**

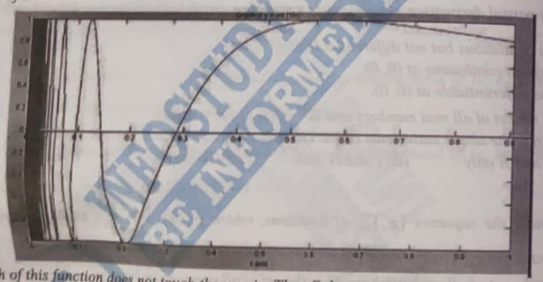
$\|f_1\|_2 = \int_0^1 t^{-2/3}dt = 3, \|f_2\|_2 = \int_0^1 t^{-4/3}dt \to \infty$.

Therefore, f_1 belongs to $L^2([0,1])$ but not f_2.

3. **Solution: (B)**

$S = \left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \le 1\right\} \cup \{(0,0)\}$

Graph of S



Graph of this function does not touch the y-axis. Thus S does not contains all the limit points $\Rightarrow$ S is not closed $\Rightarrow$ S is not compact.

$T = \left\{\left(x, x\sin\frac{1}{x}\right) : 0 < x \le 1\right\} \cup \{(0,0)\}$. Clearly T is bounded.

Also, T contains all the limit points $\Rightarrow$ T is closed.

$\Rightarrow$ option (B) is correct.

4. **Solution: (A)**

5. **Solution: (A)**

Follows from definition of uniform continuity

6. **Solution: (B)**

Clearly, $f_n \to g_2$ pointwise as $n \to \infty$, $\frac{2}{n} \to 0 \Rightarrow f_n(x) = 1$ for $0 \le x \le 1$.

But f_n does not converge uniformly as for $\in = \frac{1}{2}$ and for $x = \frac{1}{n}$,

We have $f_n\left(\frac{1}{n}\right) = 0 \ \forall n \Rightarrow \left|f_n\left(\frac{1}{n}\right) - g_2\left(\frac{1}{n}\right)\right| = 1 \not< \frac{1}{2}$

7. **Solution: (B)**

$\sum_{n=1}^{\infty} f_n(x)$ is not uniformly convergent on [0,1], but $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent on [0,1].

$\because f_n(x) = x^2(1-x^2)^{n-1}$. Let $F_n(x)$ be sequence of partial sums of $\sum_{n=1}^{\infty} f_n(x)$.

$\Rightarrow F_n(x) = x^2[1 + (1-x^2) + (1-x^2)^2 + \ldots + (1-x^2)^n] = x^2\left[\frac{1-(1-x^2)^{n+1}}{1-(1-x^2)}\right] = 1-(1-x^2)^{n+1}$

$f(x) = \lim_{n\to\infty}\left(1-(1-x^2)^{n+1}\right) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x < 1 \end{cases}$

$\therefore$ sequence function (f_n) is continuous throughout [0,1],but limit function f is discontinuous at $x = 0 \Rightarrow \sum f_n(x)$ is not uniformly convergent on [0,1]

Now, $|g_n(x)| = \left|\frac{1}{n^2(1+x^2)}\right| \le \frac{1}{n^2} \ \forall \ n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

$\Rightarrow$ By Weierstrass M-Test $\sum_{n=1} g_n \xrightarrow[on[0,1]]{uniform} g$

8. **Solution: (D)**

Clearly f is continuous function and hence f is Lipschitz continuous with Lipschitz constant 1.

$\left(as \left|\frac{f(x)-f(y)}{x-y}\right| \le 1\right)$.

$\Rightarrow \left|\frac{\partial f}{\partial x}\right| \le 1 \Rightarrow f'$ is bounded.

9. **Solution: (B)**

Given S is a non-measurable subset of $\mathbb{R}$ and T be a measurable subset of $\mathbb{R}$ such that $S \subset T$.

If $m^*(S) = 0$ then S is measurable which is a contradiction.

($\because$ Result: A set with outer measure zero is Lebesgue measurable)

$\therefore m^*(S) > 0$

If $m^*(T/S) = 0$ then S' in T is measurable, which is a contradiction.

$\therefore m^*(T/S) > 0$

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10. **Solution: (A)**

Let $c=(0,0), u=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$$f'(c,u)=\lim_{h\to 0}\frac{f(c+hu)-f(c)}{h}=\lim_{h\to 0}\frac{f\left(\frac{h}{\sqrt{2}},\frac{h}{\sqrt{2}}\right)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{h^2}{2}\left(\frac{h}{\sqrt{2}}\right)\right)}{\frac{h^4}{4}+\frac{h^2}{2}}-0=\lim_{h\to 0}\left(\frac{\frac{h^3}{2\sqrt{2}}}{\frac{h^4+2h^2}{4}}\right)\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{h^2}{2\sqrt{2}}\left(\frac{4}{h^4+2h^2}\right)=\lim_{h\to 0}\frac{1}{\sqrt{2}}\left(\frac{2}{h^2+2}\right)=\frac{1}{\sqrt{2}}$$

11. **Solution: (C)**

We know $\iint(\vec{F}.\vec{n})d\sigma=\iiint_C divF\,dxdydz$, (where C is the region of hemisphere)

$$\Rightarrow\iint(\vec{F}.\vec{n})d\sigma=\iiint 3dxdydz=3\iiint dxdydz=3\times\frac{2}{3}\pi\times(3)^3=54\pi$$

(∵ Volume of hemisphere with radius $r=\frac{2}{3}\pi r^3$)

12. **Solution: (C)**

To minimize $\int_{-1}^{1}(f(x)-p(x))^2dx$, where $p(x)=c_0+c_1x$, i.e., to minimize $\int_{-1}^{1}(f(x)-c_0-c_1x)^2dx$

i.e., to minimize $h(c_0,c_1)=\int_{-1}^{1}(f(x)-c_0-c_1x)^2dx$ w.r.t. c_0 and c_1

$$\Rightarrow \text{to minimize } h(c_0,c_1)=\int_{-1}^{1}(f(x))^2dx+2c_0^2+\frac{2}{3}c_1^2-2c_0\int_{-1}^{1}f(x)dx-2c_1\int_{-1}^{1}xf(x)dx$$

Put $\frac{\partial h}{\partial c_0}=0$, we get $c_0=\frac{1}{2}\int_{-1}^{1}f(x)dx$

$$\Rightarrow c_0=\frac{1}{2}<f,1> \quad ...(1) \quad (\because \text{Here } <f,g>=\int_{-1}^{1}fgdx)$$

Now, put $\frac{\partial h}{\partial c_1}=0$, we get $c_1=\frac{3}{2}\int_{-1}^{1}xf(x)dx=\frac{3}{2}<f,x>$ ...(2)

Hence, it can be verified that c_0 and c_1 in (1) and (2) minimizes $\int_{-1}^{1}(f(x)-p(x))^2dx$.

13. **Solution: (B)**

Here, $f(x)=x^2+x$, let $p(x)=c_0+c_1x$

$$\int_{-1}^{1}(f(x)-p(x))^2dx=\int_{-1}^{1}(x^2+x-c_0-c_1x)^2dx=\int_{-1}^{1}(x^2+(1-c_1)x-c_0)^2dx$$

$$=\int_{-1}^{1}(x^4+(1-c_1)^2x^2+c_0^2+2(1-c_1)x^3-2(1-c_1)c_0x-2c_0x^2]dx$$

$$=2\int_{0}^{1}(x^4+(1-c_1)^2x^2+c_0^2-2c_0x^2)dx$$

$$\because \int_{-a}^{a}f(x)dx=\begin{cases}0 & \text{if } f(x) \text{ is odd}\\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even}\end{cases}$$

$$=2\left[\frac{x^5}{5}+(1-c_1)^2.\frac{x^3}{3}+c_0^2x-2c_0\frac{x^3}{3}\right]_0^1=2\left[\frac{1}{5}+(1-c_1)^2.\frac{1}{3}+c_0^2-\frac{2c_0}{3}\right]=g\ (say)$$

We have to minimize $\int_{-1}^{1}(f(x)-p(x))^2dx$, i.e., to g

$$\therefore \frac{\partial g}{\partial c_0}=2(2c_0)-2\left(\frac{2}{3}\right)=0\Rightarrow c_0-\frac{1}{3}=0\Rightarrow c_0=\frac{1}{3} \text{ and } \frac{\partial g}{\partial c_1}=\frac{-4}{3}(1-c_1)=0\Rightarrow c_1=1$$

It can be verified that $c_0=\frac{1}{3}$, $c_1=1$, will minimize $\int_{-1}^{1}(f(x)-p(x))^2dx$

$$\therefore p(x)=\frac{1}{3}+x=\frac{1}{3}(1+3x)$$

14. **Solution: (A)**

As $f_1(x)=\begin{cases}\frac{1}{9}, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$, $f_2(x)=\begin{cases}\frac{1}{13}, & 0\le x\le 2\\ 0, & \text{otherwise}\end{cases}$, ..., $f_n(x)=\begin{cases}\frac{1}{4n+5}, & 0\le x\le n\\ 0, & \text{otherwise}\end{cases}$

Given $\varepsilon>0$, however small, to get $\frac{1}{4n+5}<\varepsilon$

$$\Rightarrow 4n+5>\frac{1}{\varepsilon}$$

$$\Rightarrow n>\frac{\frac{1}{\varepsilon}-5}{4}, \text{ choose } n_0\in\mathbb{N} \text{ such that } n_0>\frac{\frac{1}{\varepsilon}-5}{4} \text{ then for } n>n_0, \frac{1}{4n+5}<\varepsilon$$

$\therefore$ By definition, $f_n\to 0$ uniformly

Consider $\int_{-\infty}^{\infty}|f_n(x)|dx=\int_0^n\frac{1}{4n+5}dx=n.\frac{1}{4n+5}=\frac{1}{4+\frac{5}{n}}$

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$\therefore \lim_{n\to\infty} \int |f_n| = \frac{1}{4} \neq 0$

$\therefore f_n$ *does not converge to* 0 *in* L^1 *norm.*

15. ***Solution: (A)***

Given P_1 *and* P_2 *be two projection operators*

Consider $(P_1+P_2)^2 = (P_1+P_2)(P_1+P_2) = P_1P_1 + P_1P_2 + P_2P_1 + P_2P_2 = P_1 + P_2$

($\because$ *If* $P^2 = P$ *then P is projection operator)*

$\therefore$ P_1+P_2 *is projection if* $P_1P_2 = P_2P_1 = 0$,

No other option is projection operator.

16. ***Solution: (A)***

Here f is continuous at the point $xy \neq 0$.

$S = \mathbb{R}^2 \setminus (\{(x, y): y = 0\} \cup \{(x, y): x = 0\})$ *because* $xy = 0 \Leftrightarrow x = 0$ *or* $y = 0$

Since, $\{(x, y) : x = 0\} \cup \{(x, y): y = 0\}$ *is closed.*

$\therefore$ *S is open [* $\because$ *A set S is open iff S' is closed]*

17. ***Solution: (D)***

$\{E_n : n = 1, 2, ...\}$ *is a decreasing sequence* $E_1 \cap F = \phi$ *so* $E_n \cap F = \phi \ \forall\, n \geq 1$.

$\{E_n \cup F\}_{n\geq 1}$ *is a decreasing sequence of sets. Now* $\mu(F) = 2$, μ *is Lebesgue-measure*

$$\mu\left[\left(\bigcap_{n=1}^{\infty} E_n\right) \cup F\right] = \mu\left[\bigcap_{n=1}^{\infty} (E_n \cup F)\right]$$

$= \lim_{n\to\infty} \mu(E_n \cup F)$ *(Since* $\{E_n \cup F\}$ *is a decreasing sequence)*

$= \lim_{n\to\infty} [\mu(E_n) + \mu(F)]$ *(Since* $E_n \cap F = \phi$ *)*

$$= \lim_{n\to\infty}\left[\frac{2n+2}{3n+1} + 2\right] = \lim_{n\to\infty}\left[\frac{2+\frac{2}{n}}{3+\frac{1}{n}} + 2\right] = \frac{2}{3} + 2 = \frac{8}{3}$$

18. ***Solution: (A)***

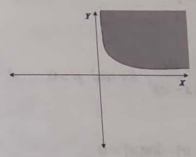
We apply Weierstrass M test to the given series

As $|\sin nx| \leq 1$

$\Rightarrow \left|\frac{1}{n^{3/2}} \sin nx\right| \leq \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}}$ *is convergent [By p-test]*

$\therefore \sum \frac{1}{n^{3/2}} \sin(nx)$ *converges uniformly on* $\mathbb{R}$.

19. ***Solution: (C)***



The value $\int_0^{\infty}\int_{1/y}^{\infty} x^4 e^{-x^2y}\, dx\, dy$

Change the order of integration $\int_{x=0}^{\infty}\left[\int_{y=1/x}^{\infty} x^4 e^{-x^2y}\, dy\right] dx = \int_{x=0}^{\infty} -xe^{-x^2y}\Big|_{1/x}^{\infty} dx$

$$= \int_0^{\infty} xe^{-x^2}\, dx = \int_0^{\infty} \frac{1}{2} e^{-t}\, dt \quad (put\ x^2 = t)$$

$$= \frac{1}{2}\left(-e^{-t}\Big|_0^{\infty}\right) = \frac{1}{2}$$

20. ***Solution: (B)***

$$\lim_{n\to\infty}\left[(n+1)\int_0^1 x^n \ln(1+x)dx\right] = \lim_{n\to\infty}\left[(n+1)\left[\frac{x^{n+1}}{n+1}\ln(1+x)\Big|_0^1 - \int_0^1 \frac{x^{n+1}}{(n+1)(1+x)}dx\right]\right]$$

$$= \lim_{n\to\infty}\left[x^{n+1}\ln(1+x)\Big|_0^1 - \int_0^1 \frac{x^{n+1}}{(1+x)}dx\right] = \ln 2 - \lim_{n\to\infty}\int_0^1 \frac{x^{n+1}}{1+x}dx$$

$\left|\frac{x^{n+1}}{1+x}\right| \leq 1$, *since* $x \in [0, 1]$

Also $\frac{x^{n+1}}{1+x} \to 0$, *as* $n \to \infty$ *(pointwise)*

So by DCT (Dominated Convergence Theorem)

$$\int_0^1 \lim_{n\to\infty}\left(\frac{x^{n+1}}{1+x}\right)dx = \int_0^1 0\ dx = 0$$

$$\Rightarrow \lim_{n\to\infty}\left[(n+1)\int_0^1 x^n \ln(1+x)dx\right] = \ln 2$$

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21. **Solution: (C)**

$f: \mathbb{R} \to \mathbb{R}$ as $f(x) = \begin{cases} x^4 & \text{if } x \text{ is rational} \\ 2x^2 - 1 & \text{if } x \text{ is irrational} \end{cases}$

Clearly, f is continuous at all points for which $x^4 = 2x^2 - 1$, i.e., $x^4 - 2x^2 + 1 = 0$

$\Rightarrow f$ is continuous at $x = -1$ and $x = 1$

22. **Solution: (C)**

Note the fact about convergence of $\int_a^b \frac{dx}{(x-a)^p}$ for real $p > 0$.

Case I: $p = 1$

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)} = \lim_{\varepsilon \to 0^+} [\log(b-a) - \log \varepsilon] = 0$$

Case II: $p \in (0,1)$

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^p} = \frac{(b-a)^{1-p}}{1-p}$$

Hence integral is convergent.

Case III: $1 < p < \infty$,

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^p} = \lim_{\varepsilon \to 0^+} \left[\frac{(b-a)^{1-p}}{1-p} - \frac{\varepsilon^{1-p}}{1-p} \right] \text{ does not exist.}$$

$(\because p > 1 \Rightarrow 1-p < 0 \Rightarrow \lim_{\varepsilon \to 0^+} \varepsilon^{1-p} = \infty)$

In our question $a = 0, b = 1$

$\therefore$ By case (II), option (C) is correct.

(Note here that in definition of $f(x), 0 \le x \le \frac{1}{n}$ does not contribute when $n \to \infty$, so in the consideration of f it is sufficient to see $\frac{1}{n} < x \le 1$)

23. **Solution: (D)**

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\left(1 - \frac{x}{4}\right) - \frac{x^5}{5!}\left(1 - \frac{x}{6}\right) - \dots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\left(\frac{4-x}{4}\right) - \frac{x^5}{5!}\frac{(6-x)}{6} - \dots - \frac{x^{2n+1}}{(2n+1)!}\left(\frac{2n+2-x}{2n+2}\right) - \dots$$

Note that, $x \in \left(\frac{1}{4}, \frac{3}{4}\right) \Rightarrow \frac{1}{4} < x < \frac{3}{4} \Rightarrow \frac{-1}{4} > -x > \frac{-3}{4}$

$\Rightarrow 4 - \frac{1}{4} > 4 - x > 4 - \frac{3}{4} \Rightarrow \frac{15}{4} > 4 - x > \frac{13}{4}$

Similarly for $n \ge 2$, $\frac{2n+2-x}{2n+2} > 0$

$\therefore e^{-x} = \sum_{j=0}^{2} \frac{(-x)^j}{j!} - \in$, where $\in > 0$

$\therefore$ (A) is not true.

Similarly, $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} \dots$

$$= \sum_{j=0}^{3} \frac{(-x)^j}{j!} + \frac{x^4}{4!}\left(1 - \frac{x}{5}\right) + \dots + \frac{x^{2n}}{2n!}\left(1 - \frac{x}{2n+1}\right) + \dots$$

Note that, $\frac{2n+1-x}{2n+1} > 0$ for $n \ge 2$

$\therefore e^{-x} = \sum_{j=0}^{3} \frac{(-x)^j}{j!} + \in$ for $\in > 0$

$\therefore$ option (B) is not true.

Appropriate pairing gives (C) also not true and (D) is true.

$\therefore$ options (A), (B) and (C) are correct and option (D) is incorrect.

24. **Solution: (A)**

$E = \{(x, y) : xy \ne 0\}$

$f: \mathbb{R} \to \mathbb{R}$, defined as $f(x, y) = \begin{cases} 0, & \text{if } xy = 0 \\ y \sin\frac{1}{x} + x \sin\frac{1}{y}, & \text{otherwise} \end{cases}$

$S_1 = \{(x, y) \in \mathbb{R}^2 \mid f_x \text{ exist}\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 \mid f_y \text{ exist}\}$ on E,

$f_x = \left(-y / x^2\right)\cos\frac{1}{x} + \sin\frac{1}{y}$, $f_y = \sin\frac{1}{x} - \frac{x}{y^2}\cos\frac{1}{y}$

$\therefore$ On E, f_x, f_y both exist

Consider, $\{(x, y) : y = 0\}$

$$f_x(x,0) = \lim_{x \to x_1} \frac{f(x,0) - f(x_1,0)}{x - x_1} = 0$$

f_x exist on $\{(x, y) : y = 0\}$

Consider, $\{(x, y) : x = 0\}$

$$f_x(0, y) = \lim_{x \to 0} \frac{f(x, y) - f(0, y)}{x} = \lim_{x \to 0} \frac{y \sin\frac{1}{x} + x \sin\frac{1}{y} - 0}{x} = \lim_{x \to 0} \frac{y \sin\frac{1}{x}}{x} + \lim_{x \to 0} \sin\frac{1}{y}$$

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But $\lim_{x\to 0}\frac{\sin\frac{1}{x}}{x}$ does not exist

$\therefore f_x$ does not exist on $\{(x,y): x=0, y\neq 0\}$

$\therefore S_1 = E \cup \{(x,y): y=0\}$

Similarly, $S_2 = E \cup \{(x,y): x=0\}$

25. Solution: (B)

$$f_x(x,y)=\begin{cases}0, & \{(x,y): y=0\}\\ \frac{-y}{x^2}\cos\frac{1}{x}+\sin\frac{1}{y}, & \text{on } E\end{cases}$$

$$f_y(x,y)=\begin{cases}0, & \{(x,y): x=0\}\\ \frac{-x}{y^2}\cos\frac{1}{y}+\sin\frac{1}{x}, & \text{on } E\end{cases}$$

Clearly f_x and f_y are continuous on E

Continuity of f_x at (x, 0)

$f_x(x,0) = 0$

$$\lim_{(x,y)\to(x,0)} f_x(x,y) = \lim_{(x,y)\to(x,0)}\left\{-\frac{y}{x^2}\cos\frac{1}{x}+\sin\frac{1}{y}\right\}$$

$$\lim_{(x,y)\to(x,0)}\frac{-y}{x^2}\cos\frac{1}{x} + \lim_{(x,y)\to(x,0)}\sin\frac{1}{y}$$

$\downarrow$ $\qquad\qquad$ $\downarrow$

0 $\qquad$ does not exist

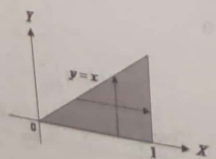
$\therefore f_x$ is not continuous at $(x, 0)$

$\therefore f_x$ is continuous only on E and $\{(0,0)\} \Rightarrow E_1 = S_1 \cap S_2 - \{(0,0)\}$

Similarly, we can check f_y is continuous only on E and $\{(0,0)\} \Rightarrow E_2 = S_1 \cap S_2 - \{(0,0)\}$

$\therefore E_1 = E_2 = S_1 \cap S_2 - \{(0,0)\}$

26. Solution: (C)



$$\iint_E (x+y)dxdy = \int_0^1\int_0^x (x+y)dydx$$

$$= \int_0^1\int_y^1 (x+y)dxdy = \int_0^1\left[\frac{x^2}{2}+xy\right]_y^1 = \int_0^1\left(\frac{1-y^2}{2}+y(1-y)\right)dy = \frac{1}{2}-\frac{1}{6}+\frac{1}{2}-\frac{1}{3}=\frac{1}{2}$$

27. Solution: (B)

$$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0$$

$$f_y(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = 0$$

Consider, $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{2xy}{x^2+y^2} = \lim_{x\to 0}\frac{2x.mx}{x^2+m^2x^2} = \frac{2m}{1+m^2}$ (for $y = mx$)

which depends on m

$\therefore$ Limit doesn't exist at (0,0) and hence 'f' is discontinuous at (0,0)

28. Solution: (C)

For option (A),

$f(-t) = f(t)$ for each $t \in [-\pi, \pi]$

$$\int_{-\pi}^{0} f(t)\,dt = -\int_{\pi}^{0} f(-u)\,du \quad [\text{putting } t = -u]$$

$$= -\int_{\pi}^{0} f(u)\,du = \int_{0}^{\pi} f(u)\,du = \int_{0}^{\pi} f(t)\,dt$$

$$\Rightarrow \int_{-\pi}^{\pi} f(t)\,dt = 2\int_{0}^{\pi} f(t)\,dt$$

For option (B),

if $f(-t) = -f(t)$

$$\int_{-\pi}^{0} f(t)\,dt = -\int_{\pi}^{0} f(-u)\,du = \int_{\pi}^{0} f(u)\,du = -\int_{0}^{\pi} f(t)\,dt$$

$$\Rightarrow \int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{0} f(t)\,dt + \int_{0}^{\pi} f(t)\,dt = -\int_{0}^{\pi} f(t)\,dt + \int_{0}^{\pi} f(t)\,dt = 0$$

Option (D) is true by First Mean Value theorem for integrals .

29. Solution: (D)

As $|f_n(x)| = \left|\frac{x^n}{n^2}\right| < \frac{1}{n^2} = M_n \quad \forall\, x\in[-1,1]$ and $\sum M_n = \sum\frac{1}{n^2}$ is convergent.

So by Weierstrass - M Test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

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30. **Solution: (B)**

Results

1. *Suppose that (i) f is continuous on $[-\pi, \pi]$*
   (ii) $f(-\pi) = f(\pi)$
   (iii) $f'(x)$ is a continuous function
   Then fourier series of f converges uniformly to f on $[-\pi, \pi]$
2. *Fourier series of a discontinuous function is not uniformly convergent at all points*

By result 1, P is true

By result 2, Q is false

31. **Solution: (A)**

$W = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4\}$

$F : W \to \mathbb{R}^3$ *is defined as*

$$F(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\vec{r}}{r^3}, \text{ where } \vec{r} = (x, y, z), |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\iint_{\partial W} F \cdot n dS = \int_W \nabla \cdot F dV = \int_W \nabla \cdot \left(\frac{\vec{r}}{r^3}\right) dV = \int_W \left[\frac{1}{r^3} \nabla \cdot \vec{r} + \vec{r} \cdot \left(-\frac{3}{r^4} \nabla r\right)\right] dV$$

$$= \int_W \left[\frac{3}{r^3} + \vec{r} \cdot \left(\frac{-3\vec{r}}{r^4 r}\right)\right] dV = \int_W \left(\frac{3}{r^3} - \frac{3\vec{r} \cdot \vec{r}}{r^5}\right) dV = \int_W \left(\frac{3}{r^3} - \frac{3r^2}{r^5}\right) dV$$

$$= \int_W \left(\frac{3}{r^3} - \frac{3}{r^3}\right) dV = \int_W 0 dV = 0$$

32. **Solution: (B)**

Dominated Convergence Theorem (DCT)

Let (f_n) be a sequence of measurable functions defined over a measurable set E such that $|f_n(x)| < \psi(x)$ $\forall x \in E$ and $\forall n \in \mathbb{N}$ where $\psi(x)$ is integrable function over E.

Let (f_n) converge in measure to a measurable function f over E.

Then $\lim_{n \to \infty} \int_E f_n(x) dx = \int_E f(x) dx$

$\therefore$ f is measurable and $\int_0^1 f_n d\mu \to 1$ *as* $n \to \infty$ *by DCT.*

33. **Solution: (C)**

34. **Solution: (B)**

As continuous image of connected set is connected and compact set is compact.

35. **Solution: (C)**

Clearly, $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ *is closed [$\because$ it contains all of its limit points]*

Also $[0,\infty)$ and $[1,2] \cup [3,4]$ are closed subsets of $\mathbb{R}$.

We know that closed subsets of a complete metric space are complete

$\therefore$ $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, *$[0,\infty)$ and $[1,2] \cup [3,4]$ are complete*

36. **Solution: (C)**

Every monotone function f is measurable because for $f^{-1}(\alpha, \infty)$ *is an interval for any* $\alpha \in \mathbb{R}$.

So (A) is not the correct answer.

Similarly (B) and (D) are not correct by definition of measurable function.

Let's give counter example to show that (C) is the right answer.

Define $f : \mathbb{R} \to \mathbb{R}$ *by following formula where E is a non – measurable set in [0, 1].*

$$f(x) = \begin{cases} 4 & \text{if } x \in (-\infty, 0) \cup (1, \infty) \\ x & \text{if } x \in E \\ x + 2 & \text{if } x \in [0,1] / E \end{cases}$$

Then f is not measurable, and $\{x \in \mathbb{R} \mid f(x) = \alpha\}$ *is measurable for each* α.

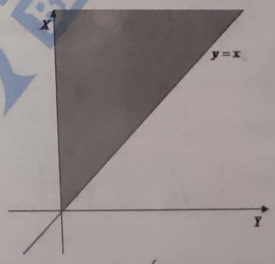
37. **Solution: (B)**

We have $f_n(x) = (1 - x)^n$

$$\xrightarrow[(\text{pointwise})]{\text{converges to}} f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \ne 0 \end{cases}, \text{ which is not continuous.}$$

Hence, the convergence is not uniform.

38. **Solution: (D)**



$$\iint_E y e^{-(x+y)} dx dy = \int_{x=0}^{\infty} e^{-x} \int_{y=x}^{\infty} y e^{-y} dy dx = \int_0^{\infty} e^{-x} \left\{\left[-y e^{-y}\right]_x^{\infty} - \int_x^{\infty} -e^{-y} dy\right\} dx = \int_0^{\infty} e^{-x} \left\{x e^{-x} -$$

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$$= \int_0^\infty e^{-x}\left[xe^{-x} - \left(0 - e^{-x}\right)\right]dx = \int_0^\infty e^{-2x}(x+1)dx$$

$$= \frac{(x+1)e^{-2x}}{-2}\Big|_0^\infty - \int_0^\infty \frac{e^{-2x}}{-2}dx = \frac{1}{2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{1}{2} + \frac{1}{2}\left[0 - \left(-\frac{1}{2}\right)\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

39. **Solution: (B)**

Theorem: *If f is real continuous function defined on closed interval $[0, 1]$, then the sequence of Bernstein polynomials $\{B_n(x)\}$, where $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} f(k/n)$ converges to f uniformly on $[0, 1]$.*

Here, $f_n(x) = \frac{1}{n}\sum_{k=0}^{n}\sqrt{k(n-k)}\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{\infty}\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}\binom{n}{k}x^k(1-x)^{n-k}$

Take, $f_1(k/n) = \sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}$

Then $f_n(x) = \sum_{k=0}^{\infty}\binom{n}{k}x^k(1-x)^{n-k} f_1(k/n)$

By theorem, $f_n(x) \to f_1(x)$ *uniformly*

$\because$ $f_n(x) \to f(x)$ *is given and sequential limit is unique*

$\therefore \lim_{n\to\infty} f_n(x) = f_1(x) = \sqrt{x(1-x)} = f(x)$

Now, f attains its maximum at x if $f'(x) = 0$ and $f''(x) < 0$

$\therefore$ *To have f maximum,* $f'(x) = 0$

$$\frac{1}{2}x^{-1/2}(1-x)^{1/2} + x^{1/2}\frac{1}{2}(1-x)^{-1/2}(-1) = 0$$

$$\Rightarrow \frac{1}{2}x^{-1/2}(1-x)^{-1/2}(1-x-x) = 0 \Rightarrow \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1-2x) = 0 \Rightarrow x = \frac{1}{2}$$

Check that $f''(x) < 0$ *at* $x = \frac{1}{2}$

$\therefore$ *f attains its maximum value at* $x = \frac{1}{2}$

$\therefore$ *Maximum value of* $f(x) = f\left(\frac{1}{2}\right) = \sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

40. **Solution: (B)**

$f(x,y) = kxy - x^3y - xy^3$

Directional derivative of f at the point $(1,2)$ in the direction of $u\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ *is given by*

$$\lim_{t\to 0}\frac{f(x+tu) - f(x)}{t-0} \Rightarrow \lim_{t\to 0}\frac{f\left((1,2) + t\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right) - f(1,2)}{t} = \frac{15}{\sqrt{2}}$$

$$\Rightarrow \lim_{t\to 0}\frac{f\left(1-\frac{t}{\sqrt{2}}, 2-\frac{t}{\sqrt{2}}\right) - f(1,2)}{t} = \frac{15}{\sqrt{2}},$$

where $f\left(1-\frac{t}{\sqrt{2}}, 2-\frac{t}{\sqrt{2}}\right) = k\left(1-\frac{t}{\sqrt{2}}\right)\left(2-\frac{t}{\sqrt{2}}\right) - \left(1-\frac{t}{\sqrt{2}}\right)^3\left(2-\frac{t}{\sqrt{2}}\right) - \left(1-\frac{t}{\sqrt{2}}\right)\left(2-\frac{t}{\sqrt{2}}\right)^3$

and $f(1,2) = (2k - 2 - 2^3)$

$$\Rightarrow \lim_{t\to 0}\frac{\left(1-\frac{t}{\sqrt{2}}\right)\left(2-\frac{t}{\sqrt{2}}\right)\left(k - \left(1-\frac{t}{\sqrt{2}}\right)^2 - \left(2-\frac{t}{\sqrt{2}}\right)^2\right) - (2k-10)}{t} = \frac{15}{\sqrt{2}}$$

$$\Rightarrow \lim_{t\to 0}\frac{\left(2-\frac{3t}{\sqrt{2}}+\frac{t^2}{2}\right)\left(k - \left(1-\frac{t}{\sqrt{2}}\right)^2 - \left(2-\frac{t}{\sqrt{2}}\right)^2\right) - (2k-10)}{t} = \frac{15}{\sqrt{2}}$$

Using L'Hospital rule and solving, we get

$$\Rightarrow 2\left(0 - 2\left(\frac{-1}{\sqrt{2}}\right) - 2.2\left(\frac{-1}{\sqrt{2}}\right)\right) + \frac{-3}{\sqrt{2}}(k-1-4) = \frac{15}{\sqrt{2}}$$

$$\Rightarrow 2\left(\frac{6}{\sqrt{2}}\right) - \frac{3}{\sqrt{2}}(k-5) = \frac{15}{\sqrt{2}} \Rightarrow \frac{12}{\sqrt{2}} - \frac{3}{\sqrt{2}}k + \frac{15}{\sqrt{2}} = \frac{15}{\sqrt{2}}$$

$\Rightarrow k = 4$

41. **Solution: (A)**

Using result, $f(x,y)$ attains a local minima at (a, b) if $f_x(a,b) = f_y(a,b) = 0$ and $f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2 > 0$ and $f_{xx}(a,b)$ or $f_{yy}(a,b) > 0$

As $f(x,y) = 4xy - x^3y - xy^3$

$\therefore$ $f_x = 4y - 3x^2y - y^3 = 0 \Rightarrow y = 0$ *or* $3x^2 + y^2 = 4$ *and* $f_y = 4x - x^3 - 3xy^2 = 0$

$\Rightarrow x = 0$ *or* $x^2 + 3y^2 = 4 \Rightarrow$ *f may attain its local minima at* $(0,0), (\pm 1, \pm 1)$

$f_{xx} = -6xy$, $f_{yy} = -6xy$, $f_{xy} = 4 - 3x^2 - 3y^2$

$\therefore$ $f_{xx}f_{yy} - f_{xy}^2 = 36x^2y^2 - (4 - 3x^2 - 3y^2)^2$

$= 36 - (4 - 3 - 3)^2$ *at* $(\pm 1, \pm 1)$

$= 36 - 4 = 32 > 0$ *and* $f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$ *at* $(0,0)$

$f_{xx}(-1,1) = 6 > 0$ *and* $f_{xx}(1,-1) = 6 > 0$

$\therefore$ *f attain its local minima at* $(-1,1)$ *and* $(1,-1)$

$\therefore f(1,-1) = -4 + 1 + 1 = -2$ *and* $f(-1,1) = -4 + 1 + 1 = -2$

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42. **Solution: (D)**

We know that two functions $f_m(x)$ and $f_n(x)$ are orthogonal if $\int_a^b f_m(x).f_n(x)\,dx = 0, \quad m \neq n$

Here, $\int_{-1}^{1} x^{2m+1}.x^{2n+1}\,dx = \int_{-1}^{1} x^{2(m+n+1)}\,dx$

$= 2\int_0^1 x^{2(m+n+1)}\,dx \quad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{when } f(x) \text{ is even}\right]$

$= 2.\left.\frac{x^{2(m+n+1)+1}}{2(m+n+1)+1}\right|_0^1 = 2\frac{1}{2(m+n+1)+1} = \frac{2}{2m+2n+3} \neq 0$

$\therefore$ Option (D) is correct.

43. **Solution: (C)**

We know that, if $f(x)$ and $g(x)$ are continuous on the interval $[a, b]$ and $f(t) \geq 0\ \forall\ t \in [a, b]$, then

$\exists\ c \in [a, b]$ such that $\int_a^b f(x)g(x)\,dx = g(c)\int_a^b f(x)\,dx$

In the given problem $f(x)$ is non-negative, i.e., $f(t) \geq 0\ \forall\ t \in [1,2]$ and $g(x) = \frac{1}{\sqrt{x}}$

$\therefore g(x) \in \left[\frac{1}{\sqrt{2}}, 1\right] \forall\ x \in [1,2] \Rightarrow k \in \left[\frac{1}{\sqrt{2}}, 1\right] \Rightarrow k \in [0.71, 1]$

$\therefore$ Option (C) is correct.

44. **Solution: (A)**

$d(x,y) = \frac{|x-y|}{1+|x-y|}$

We know that, $|x-y| < 1 + |x-y|$

$\Rightarrow \frac{|x-y|}{1+|x-y|} \leq 1 \Rightarrow d(x,y) < 1\ \forall\ x, y \in \mathbb{R}$

$\Rightarrow d(x,y)$ is bounded

We know that $(\mathbb{R}, e)$ is a metric space, where $e(x,y) = |x-y|\ \forall\ x, y \in \mathbb{R}$

Consider, $d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{e(x,y)}{1+e(x,y)}\ \forall\ x, y \in \mathbb{R}$

Then d and e are equivalent metrics.

If possible, let $(\mathbb{R}, d)$ is compact metric space then every d-open cover of $\mathbb{R}$ has a finite subcover.

Consequently, every e-open cover of $\mathbb{R}$ has a finite subcover.

$\Rightarrow$ $(\mathbb{R}, e)$ is compact, which is a contradiction.

Hence, $(\mathbb{R}, d)$ is not compact

$\Rightarrow$ option (A) is correct.

45. **Solution: (A)**

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^{\frac{5}{2}}}[1-\cos(x^2+y^2)] & ,(x,y) \neq (0,0) \\ k & ,(x,y) = (0,0) \end{cases}$$

Since $f(x,y)$ is continuous at (0, 0)

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = k$

$\Rightarrow \lim_{(x,y)\to(0,0)} \left[\frac{xy}{(x^2+y^2)^{\frac{5}{2}}}(1-\cos(x^2+y^2))\right] = k$

As $f(x,y)$ is continuous at (0, 0).

So, if we approach (0, 0) along x-axis, then the value will remain same. Then $y = 0 \Rightarrow k = 0$

$\therefore$ Option (A) is correct.

46. **Solution: (D)**

47. **Solution: (C)**

Given $\int_E f\,dm = 0$ where E is a measurable subset of $\mathbb{R}$.

Let $f(x) = 0$ be a Lebesgue measurable function

($\because$ every constant function is measurable) and $E = [0,1]$ be a measurable subset of $\mathbb{R}$

Then, $\int_E f\,dm = 0$ ($\because f = 0$) but $m(E) = 1 \neq 0$

Hence, option (A) is incorrect.

Let $f(x) = \begin{cases} 0 & x \in \mathbb{Q}^c \\ 1 & x \in \mathbb{Q} \end{cases}$ and let $E = \mathbb{R}$

Then f is measurable function and E is measurable subset of $\mathbb{R}$.

Also, $\int_E f\,dm = 0.m(\mathbb{Q}^c) + 1.m(\mathbb{Q}) = 0$ ($\because m(\mathbb{Q}) = 0$)

Now $\{x \in \mathbb{R}: f(x) = 0\} = \mathbb{Q}^c \neq E$

Hence, option (B) is incorrect.

Also, $m(\{x \in E: f(x) = 0\}) = m(\mathbb{Q}^c) \neq 0$

Hence, option (D) is incorrect.

$\therefore$ Option (C) is correct.

48. **Solution: (D)**

Take, $f_n(x) = \frac{n^2 x}{1+n^4x^2}$, $x \in [0,1]$

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$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2}=0,\ x\neq 0 \text{ and } f_n(0)=0$$

$\therefore f(x)=0\ \forall\ x\in[0,1]$

$$M_n=\sup_{x\in[0,1]}\{|f_n(x)-f(x)|\}=\sup_{x\in[0,1]}\left\{\left|\frac{n^2x}{1+n^4x^2}-0\right|\right\}=\sup_{x\in[0,1]}\left\{\left|\frac{n^2x}{1+n^4x^2}\right|\right\}\geq\frac{n^2\cdot\frac{1}{n^2}}{1+n^4\cdot\frac{1}{n^4}}\left[\text{Take } x=\frac{1}{n^2}\right]=\frac{1}{2}$$

$\therefore$ *By M_n Test $\{f_n(x)\}$ is not uniformly convergent*

Further, $\int_0^1 f(x)dx=\int_0^1 0dx=0$

$$\int_0^1 f_n(x)dx=\int_0^1\frac{n^2x}{1+n^4x^2}dx=\frac{1}{2n^2}\int_0^1\frac{(2n^4x)dx}{1+n^4x^2}=\frac{1}{2n^2}[\log(1+n^4x^2)]_0^1$$

$$=\frac{1}{2n^2}\log(1+n^4)$$

$$\therefore\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\lim_{n\to\infty}\frac{\log(1+n^4)}{2n^2}\quad\left(\frac{\infty}{\infty}\ form\right)$$

$$=\lim_{n\to\infty}\frac{\frac{1}{1+n^4}\cdot 4n^3}{4n}=\lim_{n\to\infty}\frac{n^2}{1+n^4}\left(\frac{\infty}{\infty}\right)=\lim_{n\to\infty}\frac{2n}{4n^3}=\lim_{n\to\infty}\frac{1}{2n^2}=0$$

Thus, $\int_0^1 f_n(x)dx\to\int_0^1 f(x)dx$ *as* $n\to\infty$

Clearly, $P_3\not\Rightarrow P_1$

49. ***Solution: (D)***

$$f_n(x)=\frac{x^n}{1+x},\ \therefore f(x)=\lim_{n\to\infty}\frac{x^n}{1+x}=0,\quad 0\leq x<1 \text{ and } f_n(1)=\frac{1}{2}$$

$$\therefore f(x)=\begin{cases}0, & 0\leq x<1\\ \frac{1}{2}, & x=1\end{cases}$$

Here, $x=1$ is point of discontinuity

$\therefore$ *$\{f_n(x)\}$ is not uniformly convergent*

Further, $g_n(x)=\frac{x^n}{1+nx}$

$$\therefore g(x)=\lim_{n\to\infty}\frac{x^n}{1+nx}=0,\ \forall x\in[0,1]$$

$$M_n=\sup_{x\in[0,1]}\{|g_n(x)-g(x)|\}=\sup_{x\in[0,1]}\left\{\left|\frac{x^n}{1+nx}-0\right|\right\}$$

$$=\sup_{x\in[0,1]}\left\{\left|\frac{x^n}{1+nx}\right|\right\}\geq\frac{\left(\frac{1}{n}\right)^n}{1+n.\frac{1}{n}}\left[\text{Take } x=\frac{1}{n}\right]$$

$=\frac{1}{2n^n}\to 0$ *as* $n\to\infty$

$\therefore$ *By M_n-Test, $\{g_n(x)\}$ is uniformly convergent.*

50. ***Solution: (B)***

Radius of convergence of $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ *is given by* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{R}$

$$\Rightarrow\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}\left(\frac{\sqrt{n}}{1}\right)=\frac{1}{R}\quad\Rightarrow R=1$$

$\therefore\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ *converges on* $|x|<1$

At $x=1$, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ *diverges (By p-test)*

$\therefore$ *At* $x=-1$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ *converges (By Leibniz test)*

$\therefore\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ *converges on [-1, 1)*

Now, radius of convergence of $\sum_{n=1}^{\infty}\frac{x^n}{n}$ *is given by* $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n}{n+1}=1=\frac{1}{R}$

$\therefore R=1$

$\therefore\sum_{n=1}^{\infty}\frac{x^n}{n}$ *converges on* $|x|<1$

At $x=1$, $\sum_{n=1}^{\infty}\frac{1}{n}$ *diverges*

At $x=-1$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ *converges (By Leibniz test)*

Hence, $\sum_{n=1}^{\infty}\frac{x^n}{n}$ *converges on [-1, 1)*

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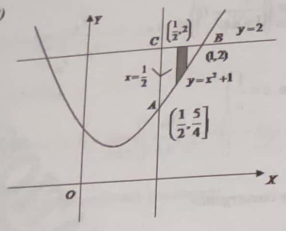
51. **Solution: (B)**



$$I=\oint_C \frac{e^y}{x}dx+\left(e^y\log x+x\right)dy \quad \ldots(i)$$

By Green's theorem, we have $\oint_C M\,dx+N\,dy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$ ...(ii)

From equation (i), we have $M=\frac{e^y}{x}, N=e^y\log x+x$

$$\frac{\partial M}{\partial y}=\frac{e^y}{x}, \frac{\partial N}{\partial x}=\frac{e^y}{x}+1 \quad \ldots(iii)$$

$$\therefore \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=1 \quad \ldots(iv)$$

So, we get $\oint_C \frac{e^y}{x}dx+\left(e^y\log x+x\right)dy$

$$=\iint_R dx\,dy=\int_{x=\frac{1}{2}}^{1}\int_{y=1/x}^{2} dy\,dx=\int_{x=\frac{1}{2}}^{1}[y]_{1/x}^{2}dx=\int_{x=\frac{1}{2}}^{1}(1-x^2)\,dx=\left(x-\frac{x^3}{3}\right)_{1/2}^{1}$$

$$=\left(1-\frac{1}{3}\right)-\left(\frac{1}{2}-\frac{1}{24}\right)=\frac{5}{24}$$

52. **Solution: (A)**

Given series is convergent if $\lim_{m\to\infty} a_m=0$, i.e., when $\lim_{m\to\infty} x^{\ln m}=0$, i.e., when $x<1$

53. **Solution: (B)**

Result: Every bounded measurable function defined on [a, b] is Lebesgue integrable function over [a, b].

54. **Solution: (B)**

If x,y,z are positive real numbers, then the minimum value of $a^3x^2+b^3y^2+c^3z^2$, where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ is at $x=\frac{a+b+c}{a}$, $y=\frac{a+b+c}{b}$, $z=\frac{a+b+c}{c}$.

Here $a=1, b=2, c=3$

$\Rightarrow x=\frac{6}{1}, y=\frac{6}{2}, z=\frac{6}{3}$

$\Rightarrow x=6, y=3$ and $z=2$

Also, $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}=\frac{1+2+3}{6}=1$ and minimum value $=(6)^2+8(3)^2+27(2)^2=36+72+108=216$

55. **Solution: (B)**

$$f_n(x)=\frac{x}{[(n-1)x+1][nx+1]}=\frac{[nx+1]-[(n-1)x+1]}{[(n-1)x+1][nx+1]}$$

$$f_n(x)=\frac{1}{[(n-1)x+1]}-\frac{1}{(nx+1)}$$

Putting $n=1, 2, 3, \ldots$

$$f_1(x)=1-\frac{1}{(x+1)}$$

$$f_2(x)=\frac{1}{(x+1)}-\frac{1}{(2x+1)}$$

$$f_3(x)=\frac{1}{(2x+1)}-\frac{1}{(3x+1)}$$

$$\vdots \quad \vdots \quad \vdots$$

$$f_{n-1}(x)=\frac{1}{(n-2)x+1}-\frac{1}{(n-1)x+1}$$

$$f_n(x)=\frac{1}{(n-1)x+1}-\frac{1}{nx+1}$$

On adding all these, we get $f_1(x)+f_2(x)+\ldots+f_n(x)=1-\frac{1}{nx+1}$, $s_n(x)=\frac{nx}{nx+1}$

Hence, $s(x)=\lim_{n\to\infty} s_n(x)=\begin{cases}0, & \text{when } x=0\\ 1, & \text{when } x\neq 0\end{cases}$

$\therefore$ Sequence function $(s_n(x))$ is continuous throughout [0,1] but limit function s(x) is discontinuous at x=0.

Hence, convergence is not uniform.

56. **Solution: (A,B)**

Dominated Convergence Theorem (DCT)

Let (f_n) be a sequence of measurable functions defined over a measurable set E such that $|f_n(x)|<\psi(x)\ \forall\ x\in E$ and $\forall\ n\in\mathbb{N}$ where $\psi(x)$ is integrable function over E.

Let (f_n) converge in measure to a measurable function f over E.

Then $\lim_{n\to\infty}\int_E f_n(x)\,dx=\int_E f(x)\,dx$

Fatou's Lemma is used to prove DCT

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Here, $s_n(x)=\frac{nx}{nx+1}$ is a sequence of measurable functions as $s_n(x)$ is a continuous function $\forall n \in \mathbb{N}$ and every continuous function is measurable.

and $|s_n(x)|=\left|\frac{nx}{nx+1}\right|<1=\psi(x)\ \forall x\in[0,1]$ and $n\in\mathbb{N}$ and $\psi(x)$ is integrable function over E.

Also $\lim_{n\to\infty} s_n(x)=s(x)=\begin{cases}0 & \text{when } x=0\\ 1 & \text{when } x\neq 0\end{cases}$

Here, s(x) is also measurable

$\therefore \lim_{n\to\infty}\int_0^1 s_n(x)\,dx=\int_0^1 s(x)\,dx=\int_0^1 dx=1$

$\therefore \lim_{n\to\infty}\int_0^1 s_n(x)\,dx=1$

57. Solution: (D)

Solving the inequality $x\sin\left(\frac{\pi}{2x}\right)\geq 0$. Since $x>0$, this yields $\sin\left(\frac{\pi}{2x}\right)\geq 0$.

$\Rightarrow$ either $\frac{1}{2}\leq x<1$ or $2k\pi\leq\frac{\pi}{2x}\leq(2k+1)\pi$ for all $k\in\mathbb{N}$

$\Rightarrow \frac{1}{2}\leq x<1$ or $\frac{1}{4k+2}\leq x\leq\frac{1}{4k}$ for all $k\in\mathbb{N}$

$\Rightarrow A=\left\{\left[\frac{1}{4k+2},\frac{1}{4k}\right]:\ k\in\mathbb{N}\right\}\cup\left[\frac{1}{2},1\right]$

Hence, $m(E)=\frac{1}{2}+\sum_{k=1}^{\infty}\left(\frac{1}{4k}-\frac{1}{4k+2}\right)=\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2k}-\frac{1}{2k+1}\right)=1-\frac{\ln(2)}{2}=1-\ln\sqrt{2}$

58. Solution: (C)

Q is $\left\{(2n+1)\frac{\pi}{2}:n\in\mathbb{Z}\right\}$ is complete because every cauchy sequence converges to a point in it.

$S=\{2n\pi:n\in\mathbb{Z}\}$. Also, $S'=\phi\subseteq S$.

So, S is closed.

59. Solution: (B)

$x, y, z > 0$

By arithmetic-geometric inequality $\frac{a}{3}=\frac{xy+xz+yz}{3}\geq\sqrt[3]{xy.xz.yz}=(f(x,y,z))^{\frac{2}{3}}$

$\Rightarrow f(x,y,z)\leq\left(\frac{a}{3}\right)^{\frac{3}{2}}$

At $x=y=z$, $f(x,y,z)=x^3$

As $x=y=z$ also satisfies the constraint. Therefore, $a=3x^2$.

Note that $f(x,y,z)=x^3=\left(\frac{a}{3}\right)^{\frac{3}{2}}$

So, maximum value is $\left(\frac{a}{3}\right)^{\frac{3}{2}}$

60. Solution: (D)

Flux across the surface S is the volume of fluid crossing S per unit time

Mathematically, Flux $=\iint_S(\vec{F}.\hat{n})\,ds$

By Divergence theorem, we have $\iint_S(\vec{F}.\hat{n})\,ds=\iiint_V div\,F\,dV$,

where V is the region enclosed by surface S.

$\therefore$ Flux of the vector field $\vec{u}=x\hat{i}+y\hat{j}+z\hat{k}$ flowing out through the surface of the ellipsoid S is

$\iint_S(\vec{u}.\hat{n})\,ds=\iiint_V 3\,dV=3\left(\frac{4}{3}\pi abc\right)=4\pi abc$

$\left(\because div\,\vec{u}=3 \text{ and volume of ellipsoid } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{4}{3}\pi abc\right)$

61. Solution: (6)

$f:\mathbb{R}\to\mathbb{R}$ continuous function with $f(1)=5$, $f(3)=11$ and $g(x)=\int_1^3 f(x+t)\,dt$

$g'(x)=f(x+t)|_1^3$

$g'(0)=f(t)|_1^3=f(3)-f(1)=11-5=6$

62. Solution: (A)

$f(x)=\sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^2}\right)$

$\Rightarrow|f(x)|=\left|\sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^2}\right)\right|\leq\sum_{n=1}^{\infty}\frac{|\sin nx|}{n^2}$

$\Rightarrow\lim_{x\to 0}|f(x)|\leq\lim_{x\to 0}\sum_{n=1}^{\infty}\frac{|\sin nx|}{n^2}=0\Rightarrow\lim_{x\to 0}|f(x)|=0\Rightarrow\lim_{x\to 0}f(x)=0$

63. Solution: (C)

For $n = 0, 1, 2, \ldots$

$x_0=0$, $x_1=1$, $x_2=\cos x_1=\cos 1$, $x_3=\cos x_2=\cos(\cos 1)$, $x_4=\cos x_3=\cos(\cos(\cos 1))$

As $x_0<x_2<x_4\ldots$ and $x_1>x_3>x_5\ldots$

So, both odd and even termed subsequences are monotonic and bounded and hence convergent.

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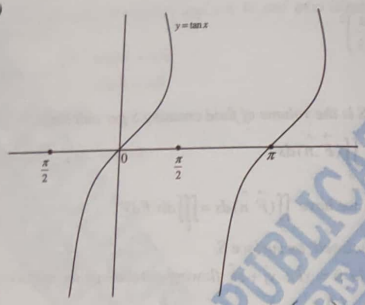
So, sequence $\{x_n\}$ is convergent.

Also, $x_{2n} < \lim_{m\to\infty} x_m < x_{2n+1} \quad \forall\, n \in \mathbb{N}$

64. ***Solution: (B)***



For the equation tan x = x, the first solution lies between $\left(0, \frac{\pi}{2}\right)$, *the second between* $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ *and so on.*

Thus, we have $a_n \in \left(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right)$

$$\Rightarrow \frac{(2n-1)\pi}{2} < a_n < \frac{(2n+1)\pi}{2}$$

$$\Rightarrow \frac{2}{(2n+1)\pi} < \frac{1}{a_n} < \frac{2}{(2n-1)\pi}$$

Now, $\sum_{n=1}^{\infty} \frac{2}{(2n+1)\pi}$ *is divergent.*

∴ By comparison test $\sum_{n=1}^{\infty} \frac{1}{a_n}$ *is divergent.*

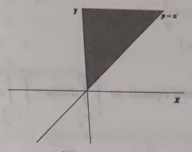
Again $b_n \in \left(\left(\frac{(2n-1)\pi}{2}\right)^2, \left(\frac{(2n+1)\pi}{2}\right)^2\right)$

$$\Rightarrow \left(\frac{2}{(2n+1)\pi}\right)^2 < \frac{1}{b_n} < \left(\frac{2}{(2n-1)\pi}\right)^2$$

Now, $\sum_{n=1}^{\infty} \left(\frac{2}{(2n-1)\pi}\right)^2$ *is convergent.*

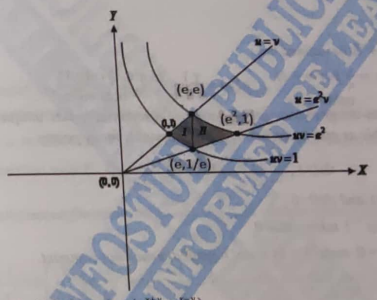
∴ By comparison test $\sum_{n=1}^{\infty} \frac{1}{b_n}$ *is convergent.*

65. ***Solution: (2)***



$$\int_0^{\infty}\int_x^{\infty} \frac{1}{y} e^{\frac{-y}{2}}\, dy dx = \int_0^{\infty}\int_0^{y} \frac{1}{y} e^{\frac{-y}{2}}\, dx dy = \int_0^{\infty} e^{\frac{-y}{2}}\, dy = -2\left(e^{\frac{-y}{2}}\right)\Big|_0^{\infty} = -2(0-1) = 2$$

66. ***Solution: (D)***



$f: \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $f(x,y) = (e^{x+y}, e^{x-y})$.

Let $u = e^{x+y}, v = e^{x-y}$

$\Rightarrow x + y = \ln u$ *and* $x - y = \ln v$

On solving, we get

$$x = \frac{\log uv}{2};\ y = \frac{1}{2}\ln\left(\frac{u}{v}\right)$$

As $0 < x < 1 \Rightarrow 0 < \ln uv < 2$

$\Rightarrow 1 < uv < e^2$

$\Rightarrow$ *Also,* $0 < y < 1 \Rightarrow 0 < \frac{1}{2}\ln\left(\frac{u}{v}\right) < 1$

$$\Rightarrow 0 < \ln\left(\frac{u}{v}\right) < 2 \Rightarrow 1 < \frac{u}{v} < e^2 \Rightarrow v < u < ve^2$$

The given region (image) is bounded by $u = v;\ u = e^2v;\ uv = 1;\ uv = e^2$

$u = v$ *and* $uv = 1 \Rightarrow u = v = 1$ *(as only positive value is considered)*

$u = v$ *and* $uv = e^2 \Rightarrow u = v = e$

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$u = e^2 v$ and $uv = 1 \Rightarrow u = e$ and $v = \frac{1}{e}$

$u = e^2 v$ and $uv = e^2 \Rightarrow u = e^2$ and $v = 1$

The shaded area is the required area and equals to $\int_1^e \int_{v=\frac{1}{u}}^{v=u} dv du + \int_e^{e^2} \int_{v=\frac{u}{e^2}}^{\frac{e^2}{u}} dv du$

$$= \int_1^e \left(u - \frac{1}{u}\right) du + \int_e^{e^2} \left(\frac{e^2}{u} - \frac{u}{e^2}\right) du = \frac{u^2}{2} - \ln u \Big|_1^e + e^2 \ln u - \frac{u^2}{2e^2} \Big|_e^{e^2}$$

$$= \frac{e^2}{2} - 1 - \frac{1}{2} + 2e^2 - \frac{e^2}{2} - \left(e^2 - \frac{1}{2}\right) = e^2 - 1$$

67. Solution: (B)

Let $f, g : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}, \quad g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then f is Riemann integrable as set of points of discontinuity has unique limit point and $g(x)$ Lebesgue integrable as $g(x)$ is discontinuous at countably many points.

68. Solution: (C)

As $x \to \infty, e^{-x} \to 0$ and $\sin 0 = 0$

So, as $x \to \infty$, $\sin(e^{-x})$ takes value 0.

So, $\{(x,y) \in \mathbb{R}^2 : y = 0, \sin(e^{-x}) = 0\}$ is not bounded, hence not compact.

69. Solution: (D)

Consider, $\frac{2^{-n^2}}{\sum_{k=n+1}^{\infty} 2^{-k^2}} > \frac{2^{-n^2}}{\sum_{k=(n+1)^2}^{\infty} 2^{-k}} = \frac{2^{-n^2}}{\frac{1}{2^{(n+1)^2}} \left(\frac{1}{1-\frac{1}{2}}\right)} = \frac{2^{-n^2} \times 2^{(n+1)^2}}{2} = 2^{2n}$

Since, $2^{2n} \to \infty$ as $n \to \infty$

$\therefore \frac{2^{-n^2}}{\sum_{k=n+1}^{\infty} 2^{-k^2}} \to \infty$ as $n \to \infty$

70. Solution: (A)

To find the supremum of $S = \{x \in \mathbb{R} : x \geq 0, \sum_{n=1}^{\infty} x^{\sqrt{n}} < \infty\}$

Let $u_n = x^{\sqrt{n}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} x^{\sqrt{n}} = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ \infty & \text{if } x > 1 \end{cases}$$

So, $S = [0,1)$. Hence, supremum of S is 1.

71. Solution: (A)

If $f \in L^1(\mathbb{R})$, then $\lim_{|x|\to\infty} f(x)$ either exists or does not exist, but if it exists it will be zero.

Also, statement 'S' is true.

72. Solution: (4)

73. Solution: (8)

74. Solution: (D)

Results:

1. Every continuous function is Lebesgue measurable function.
2. Lebesgue Dominated Convergence Theorem: Let (f_n) be a sequence of measurable function defined over a measurable set E such that $|f_n(x)| < \phi(x)\ \forall\ x \in E$ and $\forall\ n \in \mathbb{N}$ where $\phi(x)$ is integrable function over E. Then $\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx$

Since $f_n(x)$ are continuous functions on $[1, \infty)\ \forall\ n \in \mathbb{N}$ and therefore $f_n(x)$ are measurable functions (By result (1))

Also, $|f_n(x)| = \left|\frac{1}{x^2} \sin\left(\frac{1}{nx}\right)\right| < \frac{1}{x^2}$ where $\frac{1}{x^2}$ is integrable function over $[1, \infty)$

Hence by result (2), I = II

$\therefore$ Option (D) is correct.

75. Solution: (B)

Suppose $\max\{d_2, 2\}$ is a metric on X.

Let $x = y \Rightarrow \max\{d_2(x, y), 2\} = 0 \Rightarrow \max\{0, 2\} = 0$

$\Rightarrow 2 = 0$ which is contradiction.

Hence, $\max\{d_2, 2\}$ is not a metric on X.

76. Solution: (4)

Given, $f(x) = \begin{cases} -3\pi, & \text{if } -\pi < x \leq 0 \\ 3\pi, & \text{if } 0 < x < \pi \end{cases}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Since, we have to find coefficient of sin3x, i.e., b_3 only

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-3\pi)\sin nx\,dx + \int_{0}^{\pi} 3\pi\sin nx\,dx\right] = -3\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + 3\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}$$

$$= -\frac{3}{n}[-1+\cos n\pi] + \frac{3}{n}[-\cos n\pi + 1] = \left(\frac{3-3\cos n\pi}{n}\right) + \left(\frac{3-3\cos n\pi}{n}\right) = \frac{6}{n}(1-\cos n\pi)$$

$$\therefore b_3 = \frac{6}{3}(1-\cos 3\pi) = 4$$

So, the coefficient of sin 3x is 4.

77. ***Solution: (6.99 to 7.01)***

78. ***Solution: (74.99 to 75.01)***

Suppose a region G in xyz space is transformed one-to-one into the region D in uvw space, then

$$\iiint_G F(x,y,z)\,dx\,dy\,dz = \iiint_D H(u,v,w)|J(u,v,w)|\,du\,dv\,dw$$

where $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

Here, there is a change of region from g(S) to S

Hence, $\iiint_{g(S)}(2x+y-2z)\,dx\,dy\,dz = \iiint_S 2(3y+4z)+2x-3z-2(x+3y)\,|J|\,dx\,dy\,dz$

$$= \iiint_S (6y+8z+2x-3z-2x-6y).15\,dx\,dy\,dz \qquad \ldots(1)$$

$$\left(\therefore |J| = \begin{vmatrix} 0 & 3 & 4 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{vmatrix} = -3(-3)+4(6) = -9+24 = 15\right)$$

$$\therefore (1) \Rightarrow \iiint_{g(S)}(2x+y-2z)\,dx\,dy\,dz = \iiint_S (5z).15\,dx\,dy\,dz = 75\iiint_S z\,dx\,dy\,dz$$

Hence, $\alpha = 75$

79. ***Solution: (0.36 to 0.38)***

$$\iint_R xy\,dx\,dy = \int_0^1\int_{x^2}^{2-x}(xy)\,dy\,dx = \int_0^1\left(\frac{xy^2}{2}\right)_{x^2}^{2-x}dx$$

$$= \frac{1}{2}\left[\int_0^1 (x(2-x)^2 - x^5)\,dx\right] = \frac{1}{2}\left[\int_0^1 4x + x^3 - 4x^2 - x^5\,dx\right]$$

$$= \frac{1}{2}\left(\left[2x^2 + \frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^6}{6}\right]_0^1\right)$$

$$= \frac{1}{2}\left[2 + \frac{1}{4} - \frac{4}{3} - \frac{1}{6}\right] = \frac{9}{24} = 0.375$$

80. ***Solution: (C)***

Consider $f'(x) = \sin^2(x^2)$

Now $f'(x)$ is bounded function on $(0, \infty)$ and hence f is uniformly continuous on $(0, \infty)$

Also, $\lim_{x\to 0} f(x)$ *and* $\lim_{x\to 1} f(x)$ *exists and hence f is uniformly continuous on [0, 1).*

81. ***Solution: (B)***

Open sets in $(\mathbb{R}, d_1)$ are $\{(a, b) : a,b \in \mathbb{R}\}$ and any union of open intervals.

Open sets in $(\mathbb{R}, d_2)$ are $\{\{x\}: x \in \mathbb{R}\}$ and any union of singletons.

Given $f: (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$, where $f(x) = x$.

Suppose f is continuous. As $\{x\}$ is open in $(\mathbb{R}, d_2)$.

Therefore $f^{-1}(\{x\}) = \{x\}$ should be open in $(\mathbb{R}, d_1)$, but this is a contradiction as $\{x\}$ is not open in $(\mathbb{R}, d_1)$.

Hence, f is not continuous.

Let $f^{-1}: (\mathbb{R}, d_2) \to (\mathbb{R}, d_1)$, let U be an open set in $(\mathbb{R}, d_1)$.

Then $(f^{-1})^{-1}(U) = U$, which is open in $(\mathbb{R}, d_2)$.

Hence, f^{-1} is continuous.

82. ***Solution: (D)***

The limit points of the set $\left\{\left(x, \sin\frac{1}{x}\right): 0 < x \le 1\right\}$ *are $\{(0, y) : -1 \le y \le 1\}$*

Hence, neither S nor T is closed.

83. ***Solution: (A)***

Given $f_1(x) = \sum_{n=1}^{\infty}\frac{x\sin(n^2x)}{n^2}$

Let $g_n(x) = \frac{x\sin(n^2x)}{n^2}$ *and* $\sum_{n=1}^{\infty} g_n(x)$ *is a series of a function defined on [0, 1] and $\forall\, n \in \mathbb{N}$,*

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$|g_n(x)| \le \frac{1}{n^2} \quad \forall\, x \in [0,1]$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty} g_n(x)$ converges uniformly on [0, 1] (By Weierstrass M-test)

Result: If (f_n) is a sequence of continuous functions on E such that $\sum_{n=1}^{\infty} f_n = f$ and $\sum_{n=1}^{\infty} f_n$ converges uniformly on E, then f is continuous on E

Hence, by result $f_1(x)$ is continuous.

Now, consider $f_2(x) = \sum_{n=1}^{\infty} x^2 (1-x^2)^{n-1}$

Consider, $s_n(x) = \sum_{k=1}^{n} x^2 (1-x^2)^{n-1}$

$s_n(x) = 1 - (1-x^2)^n$

$\therefore\ f_2(x) = \lim_{n\to\infty} s_n(x) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x \le 1 \end{cases}$

Clearly, $f_2(x)$ is not continuous.

84. Solution: (4)

We know that $\iint_S \vec{F}.\hat{n}\, ds = \iiint_V div\, F\, dV$ ...(1)

where S is surface and V is the region enclosed by surface S and $\hat{n}$ is the unit normal outward vector

Given, $\hat{n} = (x, y, z)$ is unit normal vector

Hence $\iint_S \left\{ \left(\frac{2x}{\pi} + \sin(y^2) \right) x + \left(e^z - \frac{y}{\pi} \right) y + \left(\frac{2z}{\pi} + \sin^2 y \right) z \right\} d\sigma$

$= \iint_S \left(\left(\frac{2x}{\pi} + \sin y^2 \right) \hat{i} + \left(e^z - \frac{y}{\pi} \right) \hat{j} + \left(\frac{2z}{\pi} - \sin^2 y \right) \hat{k} \right) . \left(x\hat{i} + y\hat{j} + z\hat{k} \right) d\sigma$ ...(2)

Take $F = \underbrace{\left(\frac{2x}{\pi} + \sin y^2 \right)}_{f_1} \hat{i} + \underbrace{\left(e^z - \frac{y}{\pi} \right)}_{f_2} \hat{j} + \underbrace{\left(\frac{2z}{\pi} - \sin^2 y \right)}_{f_3} \hat{k}$

Now, $div\, F = \frac{3}{\pi} \quad \left(\because div\, F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right)$

$\Rightarrow \iint_S \vec{F}.\hat{n}\, d\sigma = \iiint_V \frac{3}{\pi} dV = \frac{3}{\pi} \times \frac{4}{3} \pi (1)^3 = 4$

85. Solution: (150)

$f(x, y)$ has extreme value at (a,b) if $f_x(a,b) = 0 = f_y(a,b)$ and $f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 > 0$.

This extreme value is maximum or a minimum according as $f_{xx}(a,b)$ or $f_{yy}(a,b)$ is negative or positive.

Then, $f_x = \frac{y}{2} - \frac{500}{x^2}$, $f_y = \frac{x}{2} - \frac{500}{y^2}$, $f_{xx} = \frac{1000}{x^3}$, $f_{yy} = \frac{1000}{y^3}$ and $f_{xy} = \frac{1}{2} = f_{yx}$

Putting $f_x = 0$ and $f_y = 0$, we get $\frac{y}{2} - \frac{500}{x^2} = 0 \Rightarrow yx^2 = 1000$ ...(i)

and $\frac{x}{2} - \frac{500}{y^2} = 0 \Rightarrow xy^2 = 1000$ ...(ii)

On solving equations (i) and (ii), we get $x = 10$, and $y = 10$

$f_{xx}(10,10) \cdot f_{yy}(10,10) - [f_{xy}(10,10)]^2 = 1.1 - \frac{1}{4} = \frac{3}{4} > 0$

Further $f_{xx}(10,10) = 1 > 0$ and $f_{yy}(10,10) = 1 > 0$

$\therefore$ (10, 10) is point of minima.

$\therefore$ Minimum value $= f(10,10) = \frac{10 \times 10}{2} + \frac{500}{10} + \frac{500}{10} = 150$

86. Solution: (D)

Consider, $f(x) = \begin{cases} x & ;\ x \in [0,1) \\ x-1 & ;\ x \in [2,3] \end{cases}$

Clearly, f(S) is connected and f is strictly increasing function.

Also, f is continuous.

87. Solution: $\left(\frac{1}{4}\right)$

$a_1 = 1,\ a_n = a_{n-1} + 4$

$\Rightarrow a_n - a_{n-1} = 4$

$\therefore \lim_{n\to\infty} \left(\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots + \frac{1}{a_{n-1} a_n} \right)$

$= \lim_{n\to\infty} \left(\frac{1}{4a_1} - \frac{1}{4a_2} + \frac{1}{4a_2} - \frac{1}{4a_3} + \ldots + \frac{1}{4a_{n-2}} - \frac{1}{4a_{n-1}} + \frac{1}{4a_{n-1}} - \frac{1}{4a_n} \right)$

$= \lim_{n\to\infty} \left(\frac{1}{4a_1} - \frac{1}{4a_n} \right) = \lim_{n\to\infty} \frac{1}{4} \left(1 - \frac{1}{4n-3} \right) = \frac{1}{4}$

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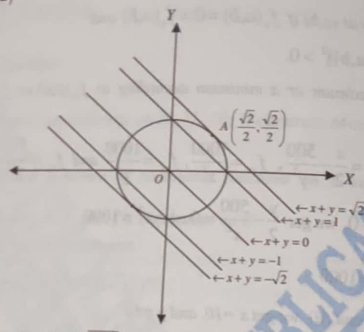
88. **Solution:** $(\sqrt{2})$



Given $S=\{x+y:(x,y)\in B(\overline{0,1})\}$

From, graph it is clear that maximum value of $x+y$ occurs at $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\in\overline{B(0,1)}$ and

maximum value of $S=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\sqrt{2}$

89. **Solution: (0.9 to 1.1)**

90. **Solution: (D)**

Given that f is a shrinking map if $|f(x)-f(y)|<|x-y|\ \forall\ x,y\in[0,1]$ ...(1)

Let $x=y$, then (1) is not satisfied. Hence, f is not a shrinking map

Also, given that f is a contraction map if there exist $\alpha<1$ such that

$|f(x)-f(y)|\le\alpha|x-y|\ \forall\ x,y\in[0,1]$

Now, if f is contraction map then $\exists\ \alpha<1$ such that $|f(x)-f(y)|\le\alpha|x-y|<|x-y|\ \forall\ x,y\in[0,1]$

Hence, f is contraction map

$\Rightarrow f$ is a shrinking map, which is a contradiction.

Therefore, f is not a contraction map.

91. **Solution: (1.9 to 2.1)**

Given $f(x)=x^2+4\sin x\cos x$

Now, fourier coefficients are given by $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$ and

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

Now, on computing we get $a_0=\frac{\pi^2}{3}$, $a_n=\frac{4}{n^2}(-1)^n$ and $b_2=2$ and $b_n=0\ \forall\ n\neq 2$

$$\therefore\left|\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n\right|=\left|\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n4}{n^2}-2\right|=\left|\frac{\pi^2}{3}-\frac{4\pi^2}{12}-2\right|\quad\left(\because\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}\right)$$

$=|-2|=2$

92. **Solution: (A)**

$S_n=\sum_{k=1}^{n}\frac{1}{k}$ and $I_n=\int_1^n\frac{x-[x]}{x^2}dx$

$$S_{10}=\sum_{k=1}^{10}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{10}\qquad\ldots(1)$$

$$I_{10}=\int_1^{10}\frac{x-[x]}{x^2}dx=\int_1^{10}\frac{1}{x}dx-\int_1^{10}\frac{[x]}{x^2}dx=(\ell nx)_1^{10}-\int_1^{10}\frac{[x]}{x^2}dx$$

$$=\ell n10-\ell n1-\int_1^{10}\frac{[x]}{x^2}dx=\ell n10-\int_1^{10}\frac{[x]}{x^2}dx\qquad\ldots(2)$$

Let $I_{10}=\ell n\,10-A$, where $A=\int_1^{10}\frac{[x]}{x^2}dx=\int_1^2\frac{[x]}{x^2}dx+\int_2^3\frac{[x]}{x^2}dx+\ldots+\int_9^{10}\frac{[x]}{x^2}dx$

$$=\int_1^2\frac{1}{x^2}dx+2\int_2^3\frac{1}{x^2}dx+\ldots..+9\int_9^{10}\frac{1}{x^2}dx=\left.\frac{-1}{x}\right|_1^2+2\left.\frac{-1}{x}\right|_2^3+\ldots+9\left.\frac{-1}{x}\right|_9^{10}$$

$$=-\left(\frac{1}{2}-1\right)-2\left(\frac{1}{3}-\frac{1}{2}\right)-3\left(\frac{1}{4}-\frac{1}{3}\right)+\ldots-9\left(\frac{1}{10}-\frac{1}{9}\right)$$

$$A=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{9}-\frac{9}{10}$$

from (2), $I_{10}=\ell n\,10-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\ldots-\frac{1}{9}+\frac{9}{10}$

$$\therefore\ S_{10}+I_{10}=\ell n\,10+\frac{9}{10}+\frac{1}{10}=\ell n\,10+1$$

93. **Solution: (0.9 to 1.1)**

94. **Solution: (-0.1 to 0.1)**

Given, $f(x)=\left(\int_0^x e^{-t^2}dt\right)^2$

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Now, $f'(x)=2\left(\int_0^{x^2} e^{-t^2}dt\right)e^{-x^2}=2e^{-x^2}\int_0^{x^2} e^{-t^2}dt$ ...(1)

Given, $g(x)=\int_0^1 \frac{e^{-x^2}(1+t^2)}{1+t^2}dt$

$g'(x)=\int_0^1\left(\frac{e^{-x^2}(1+t^2)}{1+t^2}\right)(-2x(1+t^2))dt=\int_0^1 -2x\ e^{-x^2(1+t^2)}dt$

$=-2x\,e^{-x^2}\int_0^1 e^{-x^2t^2}dt=-2xe^{-x^2}\int_0^x e^{-u^2}\frac{du}{x}$ (Put $xt=u$)

$=-2e^{-x^2}\int_0^x e^{-u^2}du$ (Change $u=t$)

$=-2e^{-x^2}\int_0^x e^{-t^2}dt$ ...(2)

Now adding (1) and (2) we get $f'(x)+g'(x)=0$

$\Rightarrow f'(\sqrt{\pi})+g'(\sqrt{\pi})=0$

95. Solution: (D)

Given that $f_n:[0,1]\to\mathbb{R}$ be defined by $f_n(x)=\frac{2x^2}{x^2+(1-2nx)^2}, n=1,2,3,...$

$\therefore \lim_{n\to\infty} f_n(x)=f(x)=0,\ \forall x\in[0,1]$

$\Rightarrow f_n(x)$ converges pointwise to 0 on $[0,1]$

$\Rightarrow$ option (C) is incorrect.

For uniform convergence

We define $M_n=\sup_{x\in[0,1]}|f_n(x)-f(x)|=\sup_{x\in[0,1]}\left|\frac{2x^2}{x^2+(1-2nx)^2}-0\right|=\sup_{x\in[0,1]} g(x)$, where $g(x)=\frac{2x^2}{x^2+(1-2nx)^2}$

Now $g'(x)=\frac{[x^2+(1-2nx)^2](4x)-(2x^2)[2x+(1-2nx)(2)(-2n)]}{[x^2+(1-2nx)^2]^2}$

$=\frac{4x(1-2nx)}{[x^2+(1-2nx)^2]^2}$

$g'(x)=0\ \Rightarrow\ x=\frac{1}{2n}$

$\Rightarrow$ At $x=\frac{1}{2n}$, $g(x)$ has maxima $\left[g''(x)<0 \text{ at } x=\frac{1}{2n}\right]$

$M_n=\sup_{x\in[0,1]} g(x)=g\left(\frac{1}{2n}\right)=\frac{2\left(\frac{1}{2n}\right)^2}{\left(\frac{1}{2n}\right)^2+\left(1-2n.\frac{1}{2n}\right)^2}=2$

Since $\lim_{n\to\infty} M_n=2\neq 0\ \Rightarrow f_n$ does not converges uniformly on $[0,1]$.

$\Rightarrow f_n$ does not have a subsequence that converges uniformly on $[0,1]$

$\therefore$ Option (D) is correct.

96. Solution: (36)

$\frac{1}{\pi}\oint_C\left(3y-e^{\cos x^2}\right)dx+\left(7x+\sqrt{y^4+11}\right)dy=?$

$C=\{(x,y)\in\mathbb{R}^2: x^2+y^2=9\}$

$\oint_C Mdx+Ndy=\iint_D\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy,$

where D is region bounded by positively oriented, piecewise smooth simple closed curve C in a plane

Here $N=7x+\sqrt{y^4+11}$ and $M=3y-e^{\cos x^2}$

$\frac{\partial N}{\partial x}=7$ and $\frac{\partial M}{\partial y}=3\ \Rightarrow\frac{1}{\pi}\iint_D(7-3)\,dx\,dy=\frac{4}{\pi}\iint_D dx\,dy=\frac{4}{\pi}$ (area of circle $x^2+y^2=9$)

$=\frac{4}{\pi}(9\pi)=36$ [$\because$ area of a circle of radius $r=\pi r^2$]

97. Solution: (B)

For Statement (Q)

Consider, $f(x)=1\ \forall x\in\mathbb{R}$ and $g(x)=\begin{cases}1 & x\in\mathbb{Q}^c\\ 0 & x\in\mathbb{Q}\end{cases}$

$\therefore m\{x: f(x)\neq g(x)\}=m(\mathbb{Q})=0$

$\therefore f=g$ almost everywhere on $\mathbb{R}$

But g is discontinuous on $\mathbb{R}$.

Hence, statement (Q) is not true.

98. Solution: (D)

Given $c_n=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{i(n-i)x}dx\ \Rightarrow\ c_n=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^x e^{inx}\,dx$

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$\Rightarrow c_n$ represents fourier coefficients w.r.t. function $f(x) = e^x$

Also we have $\sum_{n=1}^{\infty} |c_n|^2 = \| f \|^2$ (By Parseval's Identity)

Hence, $\| f \|^2 = \| e^x \|^2 = \int_{-\pi}^{\pi} e^x \cdot e^x \, dx = \int_{-\pi}^{\pi} e^{2x} dx = \frac{e^{2\pi} - e^{-2\pi}}{2} = \sin h(2\pi)$

99. Solution: (B)

Let $y_n = \sum_{i=1}^{n} \frac{1}{i}$ and let $(\mathbb{R}, d)$ be a complete metric space where d is the usual metric.

Then $d(y_n, y_{n+1}) = \left| \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n+1} \frac{1}{n+1} \right| = \left| \frac{1}{n+1} \right| < \frac{1}{n} \quad \forall \; n \in \mathbb{N}$

But (y_n) is not convergent.

Hence, (y_n) need NOT converge in complete metric soace (X, d)

100. Solution: (9)

101. Solution: (C)

Given $f(x, y) = \begin{cases} \sin\left(\frac{y^2}{x}\right)\sqrt{x^2 + y^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Let $\epsilon > 0$ be given

Now, $|f(x, y) - f(0,0)| = \left| \sin\left(\frac{y^2}{x}\right)\sqrt{x^2 + y^2} - 0 \right| = \left| \sin\left(\frac{y^2}{x}\right) \right| \left| \sqrt{x^2 + y^2} \right| \leq \sqrt{x^2 + y^2} < \epsilon$

or if $x^2 + y^2 < \epsilon^2$

$\Rightarrow x^2 < \frac{\epsilon^2}{2}, y^2 < \frac{\epsilon^2}{2} \Rightarrow |x| < \frac{\epsilon}{\sqrt{2}}, |y| < \frac{\epsilon}{\sqrt{2}}$

$\therefore |f(x, y) - f(0, 0)| < \epsilon$, whenever $|x-0| < \delta$, $|y-0| < \delta$ where $\delta = \frac{\epsilon}{\sqrt{2}} > 0$

$\Rightarrow \lim_{(x,y) \to (0,0)} f(x, y) = f(0,0)$ and hence f is continuous at $(0,0)$

Now, $f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \frac{\sin\left(\frac{k^2}{0}\right)\sqrt{0 + k^2} - 0}{k}$ which does not exist

$\Rightarrow f$ is not differentiable at (0, 0).

Now, let $u = (u_1, u_2)$ be any unit vector

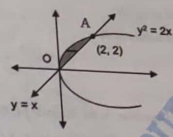
$\therefore$ Directional derivative of f at (0, 0) in the direction of u is

$\lim_{h \to 0} \frac{f(u_1 h, u_2 h) - f(0,0)}{h} = \lim_{h \to 0} \sin\left(\frac{u_2^2 h}{u_1}\right)\sqrt{u_1^2 + u_2^2} = 0$ which exist for every (u_1, u_2)

Hence, f has directional derivative in all directions

$\therefore$ Option (C) is correct.

102. Solution: (2)



D is the given region bounded by $y^2 = 2x$ and $y = x$

Then $\iint_D 3xy \, dx \, dy = ?$

The point of intersection of $y^2 = 2x$ and $y = x$ are O(0, 0) and A(2, 2)

Taking horizontal strips, x varies from $\frac{y^2}{2}$ to y and y varies from 0 to 2

$$I = \iint_D 3xy \, dx \, dy = 3\int_0^2 y \left(\int_{\frac{y^2}{2}}^{y} x \, dx \right) dy = 3\int_0^2 y \left(\frac{x^2}{2} \Big|_{\frac{y^2}{2}}^{y} \right) dy = \frac{3}{2}\int_0^2 y \left[y^2 - \frac{y^4}{4} \right] dy$$

$$= \frac{3}{2}\int_0^2 \left(y^3 - \frac{y^5}{4} \right) dy = \frac{3}{2}\left[\left(\frac{y^4}{4} \right)_0^2 - \frac{1}{4}\left(\frac{y^6}{6} \right)_0^2 \right] \Rightarrow I = \frac{3}{2}\left[4 - \frac{1}{24} 2^6 \right] = \frac{3}{2}\left[4 - \frac{8}{3} \right] = 6 - 4 = 2$$

103. Solution: (B)

$f: X \to Y$ be a continuous where X and Y are metric spaces

For option (A),

Define $f: \mathbb{R} \to \mathbb{R}$ as $f(x) = 1$ and also f is continuous map

Take $S = \mathbb{R} \subseteq \mathbb{R}$ (open subset of $\mathbb{R}$) but $f(S) = \{1\}$ which is closed in $\mathbb{R}$

So, option (A) is not true

For option (B),

Continuous image of connected set is always connected set

So option (B) is true.

For option (c),

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Define $f: \mathbb{R} \to \mathbb{R}$ as $f(x) = e^{-x}$ which is continuous map

Take $S = [0, \infty) \subseteq \mathbb{R}$ (closed subset of $\mathbb{R}$)

But $f(S) = (0,1]$ which is not closed subset of $\mathbb{R}$

So, option (C) is not true.

For option (D),

Define $f: (0,1] \to \mathbb{R}$ as $f(x) = \frac{1}{x}$ which is continuous map

Take $S = (0,1] \subseteq (0,1]$ (bounded subset of $(0,1]$)

But $f(S) = [1, \infty)$ which is not bounded subset of $\mathbb{R}$

So, option (D) is not true.

104. Solution: (1)

Since $I = [2, 3]$, J be the set of rational numbers in the interval $[4, 6]$

i.e., $J = [4, 6] \cap \mathbb{Q}$ and K be the Cantor Set and $L = \{7 + x \mid x \in K\}$

And $m^*(I \cup J \cup L) = m^*(I) + m^*(J) + m^*(L)$

$m^*(I)$ = length of interval $[2, 3] = 1$

$m^*(J) = 0$ [$\because [4, 6] \cap \mathbb{Q}$ is countable and measure of countable set is zero]

$m^*(L) = 0$ [$\because m^*(7 + K) = m^*(K)$ and Cantor set has measure zero]

$\Rightarrow m^*(I \cup J \cup L) = 1 + 0 + 0 = 1$

105. Solution: (6)

$u(x,y,z) = x^2 - 2y + 4z^3$

The directional derivative of 'u' in direction of $\vec{a} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{k}$ at the point $P(5,1,0)$ is given

$$\lim_{t\to 0}\frac{u(P+t\vec{a})-u(P)}{t} = \lim_{t\to 0}\frac{u\left((5,1,0)+t\left(\frac{3}{5},0,\frac{-4}{5}\right)\right)-u(5,1,0)}{t} = \lim_{t\to 0}\frac{u\left(5+\frac{3t}{5},1,\frac{-4t}{5}\right)-u(5,1,0)}{t}$$

$$= \lim_{t\to 0}\frac{\left(5+\frac{3t}{5}\right)^2 - 2(1) + 4\left(\frac{-4t}{5}\right)^2 - 23}{t} = \lim_{t\to 0}\frac{25+\frac{9t^2}{25}+6t-2+\frac{64t^2}{25}-23}{t} = \lim_{t\to 0}\frac{9t}{25}+6+\frac{64t}{25} = 6$$

So, answer is 6.

Shortcut method:

The directional derivative of u in direction of $\vec{a}$ at point P is given by

$$(\nabla u.\vec{a})(p) = \left(\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right)\cdot\vec{a}\right) \text{ at } (5,1,0)$$

$$= \left(2x\hat{i} - 2\hat{j} + 8z\hat{k}\right)\cdot\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{k}\right) \text{ at } (5,1,0)$$

$$= \left(\frac{6}{5}x - \frac{32}{5}z\right) \text{ at } (5,1,0) = 6$$

106. Solution: (A)

$D = \{(x, y) : x^2 + y^2 \le 2\}$, put $x = r\cos\theta$ and $y = r\sin\theta$

$$\iint_D e^{-(x^2+y^2)}dxdy = \int_0^{2\pi}\int_0^{\sqrt{2}} e^{-r^2}\, r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{2}} e^{-r^2}\, 2r\,dr\,d\theta$$

Put $r^2 = t$ so that $2rdr = dt$ (when $r = 0 \Rightarrow t = 0$ and when $r = \sqrt{2} \Rightarrow t = 4$)

$$= \frac{1}{2}\int_0^{2\pi}\left(\int_0^4 e^{-t}dt\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left[\frac{e^{-t}}{-1}\right]_0^4 d\theta = \frac{1}{2}\int_0^{2\pi}[-e^{-4}+1]d\theta = \frac{1}{2}(1-e^{-4})\int_0^{2\pi}d\theta = \frac{1}{2}(1-e^{-4})2\pi = \pi(1-e^{-4})$$

$\therefore$ Option (A) is correct.

107. Solution: (18)

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV \quad \text{(Using Gauss Divergence Theorem)}$$

where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x\hat{i} + y\hat{j} + z\hat{k})\,dV$$

$$= 3.\int_0^1\int_0^2\int_0^3 dz\,dy\,dx = 3(1-0)(2-0)(3-0) = 18$$

108. Solution: (2)

As we know that if $f(x)$ is function define on (a,b) and periodic on (a, b), then its fourier series is given by $f(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left[a_n'\cos\left(\frac{n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n'\sin\left(\frac{n\pi x}{b-a}\right)\right]$ ...(1)

where $a_0' = \frac{2}{b-a}\int_a^b f(x)dx$

$$a_n' = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{n\pi x}{b-a}\right)dx$$

$$b_n' = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{n\pi x}{b-a}\right)dx$$

If $f(x) = \sin(\pi x)$, $0 < x < 1$

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$$a_0' = \frac{2}{1-0}\int_0^1 \sin(\pi x)dx = 2\left[\frac{-\cos \pi x}{\pi}\right]_0^1 = \frac{4}{\pi}$$

$$a_1' = \frac{2}{1}\int_0^1 \sin(\pi x)\cos(\pi x)\,dx = \int_0^1 \sin(2\pi x)\,dx$$

$$a_1' = 0$$

And given that $\sin \pi x = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x) + b_n \sin(n\pi x)]$ ...(2)

for $0 < x < 1$, *and* $(a_0 + a_1) = ?$

Comparing (1) and (2) we get $a_0 = \frac{a_0'}{2} = \frac{4}{2\pi} = \frac{2}{\pi}$ *and* $a_1 = a_1' = 0$

$\Rightarrow a_0 = \frac{2}{\pi}$ *and* $a_1 = 0$

$\therefore (a_0 + a_1)\pi = \left(\frac{2}{\pi} + 0\right)\pi = 2$ *and hence* $(a_0 + a_1)\pi = 2$

109. Solution: (1)

Given $f_n(x) = \frac{2n\,x^{n-1}}{1+x}$, $x \in [0,1]$

Let $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \ell$ $(\ell \in \mathbb{R})$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 \frac{2nx^{n-1}}{1+x}dx = \ell \qquad ...(1)$$

Now, $\frac{x^{n-1}}{1+x} = \frac{x^n}{x(1+x)} = x^n\left[\frac{1}{x} - \frac{1}{1+x}\right] = x^{n-1} - \frac{x^n}{1+x}$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 \frac{2n\,x^{n-1}}{1+x}dx = \lim_{n\to\infty}\int_0^1 2n\left[x^{n-1} - \frac{x^n}{1+x}\right]dx = \lim_{n\to\infty}\left[2n\int_0^1 x^{n-1}dx - 2n\int_0^1 \frac{x^n}{1+x}dx\right]$$

$$= \lim_{n\to\infty}\left[2n\left[\frac{x^n}{n}\right]_0^1 - 2(n+1)\int_0^1 \frac{x^n}{1+x}dx + 2\int_0^1 \frac{x^n}{1+x}dx\right]$$

$$= \lim_{n\to\infty}[2 - \ell + 2(0)] \qquad \left[\text{from (1) and } \int_0^1 \frac{x^n}{1+x}dx = 0 \text{ By DCT}\right]$$

$= 2 - \ell$ ...(2)

(1) and (2) $\Rightarrow \ell = 2 - \ell$

$\Rightarrow 2\ell = 2 \Rightarrow \ell = 1$

110. Solution: (C)

$$u_n = \int_1^n e^{-x^2}dt = \int_0^n e^{-x^2}dt - \int_0^1 e^{-x^2}dt$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\int_0^n e^{-x^2}dt - \int_0^1 e^{-x^2}dt = \int_0^{\infty} e^{-x^2}dt - \int_0^1 e^{-x^2}dt = \frac{\sqrt{\pi}}{2} - \text{finite quantity} = \ell \text{ (say)}, \text{ where } \ell \text{ is non-zero finite value.}$$

Hence the sequence $\{u_n\}_{n=1}^{\infty}$ *is convergent*

As, $\lim_{n\to\infty} u_n = \ell \neq 0 \Rightarrow \sum_{n=1}^{\infty} u_n$ *is not convergent because if a series* $\sum_{n=1}^{\infty} u_n$ *is convergent, then* $\lim_{n\to\infty} u_n$ *must be zero*

$\therefore$ *option (C) is correct.*

111. Solution: (C)

Take $\phi = 0$, *i.e.,* $\phi(x, y, z) = 0\ \forall\ x, y, z \in \Gamma$

Now $\vec{\nabla}\phi = 0$, *which is solenoidal as well as irrotational.*

So, options (A), (B) and (D) are incorrect.

$\therefore$ *option (C) is correct.*

112. Solution: (D)

$X = \{(x_1, x_2, \ldots): x_i \in \mathbb{R}$ *and only finitely many* x_i*'s are non- zero*$\}$ *and*

$d = \sup_{i\in\mathbb{N}} |x_i - y_i|$ *for* $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ *in X.*

X consists of eventually zero sequence.

For statement P: X is not a closed subspace with respect to the given metric

For example, the sequence (x_n) *where* x_n *has* $\frac{1}{k}$ *for the first n entries and is zero elsewhere,*

i.e., $x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$ *is Cauchy w.r.t given metric but not convergent (to a sequence in X).*

Let C_0 *be a space consisting of all sequences converging to 0, i.e., null sequences.*

C_0 *is a closed and complete space w.r.t the given metric*

Clearly, X is a subspace of C_0 *and A subspace of a complete metric space is complete if it is closed as X is not closed, hence it is not a complete metric space w.r.t the given metric.*

For statement Q: The set $\{x \in X: d(0,x) \leq 1\}$.

It is easy to show that this set is not closed by taking the same example which has been taken for statement 'P'.

As it is not closed, hence it is not compact also.

$\therefore$ *Option (D) is correct.*

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113. Solution: (B)

Since, $\mathbb{Q}$ and $\mathbb{Z}$ both are countable sets and finite product of two countable sets is countable.

$\therefore \mathbb{Q} \times \mathbb{Z}$ is countable

Let $S = \{f : f$ is a function from $\mathbb{N}$ to $\{0, 1\}$.

$|S| = 2^{\aleph_0} \Rightarrow S$ is uncountable

Let the set $\{\sqrt{p} : p$ is a prime number$\} = R$ and let its elements are in increasing order $\{\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \ldots\}$

Let $f : \mathbb{N} \to R$ be defined by $f(n) = \sqrt{p_n}$, where p_n denotes n^{th} prime.

Clearly, the function is one-one as well as onto.

$\therefore R$ is countable

Let us denote infinite set as $T = \{a_1, a_2, a_3, a_4, \ldots\}$

Let $T' = \{a_1, a_2, a_3, a_4, \ldots\}$ be a proper subset of T.

Let $f : T \to T'$ be defined as $f(a_n) = a_{n+1} \; \forall \; n \geq 0$

Clearly, f is a bijection from T to T'.

Hence, Statements II, IV are correct while I, III are incorrect

$\therefore$ option (B) is correct.

114. Solution: (C)

Given, $f(x,y) = x^6 - 2x^2y - x^4y + 2y^2 \Rightarrow f_x = 6x^5 - 4xy - 4x^3y$ and $f_y = -2x^2 - x^4 + 4y$

Put $f_x = 0$ and $f_y = 0$ we get that (0,0) is a critical point.

Now, $f_{xx}(0,0) = 30(0) - 4(0) - 12(0) = 0$, $f_{yy}(0,0) = 4$

$f_{xy}(0,0) = -4(0) - 4(0) = 0 \Rightarrow (f_{xx} f_{yy} - f_{xy}^2)(0,0) = 0$

Hence, (0,0) is a saddle point.

$\therefore$ option (C) is correct.

115. Solution: (A)

Let $a_n = \frac{1}{n}$ and $\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$

Radius of convergence of $\sum_{n=0}^{\infty} \frac{1}{n} x^n$ is given by $r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{n+1}{n} = 1$

Hence, options (B), (C) and (D) are incorrect.

So, option (A) is correct.

116. Solution: (C)

$g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $g(x, y) = (e^x \cos y, e^x \sin y)$

$g(0, 0) = (1, 0)$ and $g(0, 2\pi) = (1, 0)$,

$g(0, 0) = g(0, 2\pi)$ but $(0, 0) \neq (0, 2\pi)$

$\therefore g$ is not injective.

For $(0,0) \in \mathbb{R}^2$, $\nexists$ any $(x, y) \in \mathbb{R}^2$ such that $g(x, y) = (0, 0)$

$\therefore g$ is not surjective.

Now, $(a,b) = g\left(1, \frac{\pi}{3}\right) = \left(e\cos\frac{\pi}{3}, e\sin\frac{\pi}{3}\right) = \left(\frac{e}{2}, \frac{e\sqrt{3}}{2}\right)$

If $h(x, y)$ is inverse of g in some nbhd of (a, b), then $h(x, y) = \left(\frac{1}{2}\log(x^2 + y^2), \tan^{-1}\frac{y}{x}\right) = (h_1, h_2)$

Now, Jacobian of $h = \begin{vmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$

At $(a,b) = \left(\frac{e}{2}, \frac{\sqrt{3}e}{2}\right)$, Jacobian $= \frac{1}{\frac{e^2}{4} + \frac{3e^2}{4}} = \frac{1}{e^2} = e^{-2}$

$\therefore$ option (C) is correct.

117. Solution: (Ans: 0)

Given that, $u_n = \frac{n!}{1.3.5\ldots(2n-1)}, n \in \mathbb{N}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{1.3.5\ldots(2n-1)(2n+1)} \cdot \frac{1.3.5\ldots(2n-1)}{n!} = \lim_{n\to\infty} \frac{(n+1)}{2n+1} = \frac{1}{2} < 1$

Hence, by ratio test $\lim_{n\to\infty} u_n = 0$.

118. Solution: (Ans: -7.5)

Let $f(x) = x^2$

$\because \frac{df}{dx} = a f(x) + 2 f(x+1) + b f(x+2) \; \forall \; x \in \mathbb{R}$

$\therefore 2x = ax^2 + 2(x+1)^2 + b(x+2)^2$

By comparing the coefficients of like terms, we get $a + 2 + b = 0$, $2 + 4b = 0$

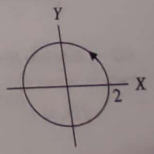
$\Rightarrow b = -\frac{1}{2}, a = \frac{-3}{2}$

So, $4a + 3b = -\frac{15}{2} = -7.5$.

119. Solution: (Ans: 32)

By Green's theorem in plane, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$

$\oint_C (3x - 4x^2 y)\, dx + (4xy^2 + 2y)\, dy = \iint_R (4y^2 - (-4x^2))\, dx\, dy = 4 \iint_R (x^2 + y^2)\, dx\, dy$



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Converting to polar form, we get $=4\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\,dr\,d\theta = 4.\frac{r^4}{4}\Big|_0^2 .\theta\Big|_0^{2\pi} = 32\pi$

$\therefore \frac{L}{\pi} = 32.$

120. Solution: (Ans: 0.22)

$T(x,y,z) = \frac{K}{x^2+y^2+z^2}$

Given, $T(0,0,1) = \sqrt{3} \Rightarrow K = \sqrt{3}$

$\therefore T(x,y,z) = \frac{\sqrt{3}}{x^2+y^2+z^2}$

Now $\overrightarrow{QR} = -\hat{i}-\hat{j}-\hat{k}$

Unit Vector in direction of $\overrightarrow{QR} = \frac{1}{\sqrt{3}}(-\hat{i}-\hat{j}-\hat{k})$

Now, $\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} = -2\sqrt{3}\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^2}$

Now, $\nabla T|_{(1,1,2)} = \frac{-2\sqrt{3}}{36}(\hat{i}+\hat{j}+2\hat{k})$

Rate of change of T in direction of $\overrightarrow{QR} = \nabla T.\hat{QR}$

$= \frac{-2\sqrt{3}}{36}(\hat{i}+\hat{j}+2\hat{k}).\frac{-1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k}) = \frac{1}{18}(1+1+2) = \frac{4}{18} = 0.22.$

121. Solution: (Ans: 0.5)

Given that $\int_0^b \int_0^{f(x)} dy\,dx = \sqrt{3+b^2} - \sqrt{3}$

$\int_0^b f(x)\,dx = \sqrt{3+b^2} - \sqrt{3}$

Differentiating both sides w.r.t. b, we get $f(b) = \frac{b}{\sqrt{3+b^2}}$

$\therefore f(1) = \frac{1}{2} = 0.5.$

122. Solution: (B)

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h^2}-0}{h} = \lim_{h\to 0} h\sin\frac{1}{h^2} = 0$

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{k^2\sin\frac{1}{k^2}-0}{k} = \lim_{k\to 0} k\sin\frac{1}{k^2} = 0$

but $\lim_{(x,y)\to(0,0)} f_x = \infty$ and $\lim_{(x,y)\to(0,0)} f_y = \infty$

Hence, statement (I) is correct.

Also $f(h,k) - f(0,0) = (h^2+k^2)\sin\left(\frac{1}{h^2+k^2}\right)$

$= \sqrt{h^2+k^2}\left(\sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right)\right)$

$\lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = \lim_{r\to 0} r\sin\left(\frac{1}{r^2}\right) = 0$

Hence, f is differentiable at (0,0).
Hence, (IV) is correct and (II) is incorrect.
Hence, option (B) is correct.

123. Solution: (D)

Given, $g_n(x) = \frac{x}{1+nx^2}, x \in \mathbb{R}$

$\lim_{n\to\infty} g_n(x) = g(x) = 0 \ \forall\, x \in \mathbb{R}$

By M_n-test,

$M_n = \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| = \frac{1}{2\sqrt{n}}$ and $\lim_{n\to\infty} M_n = \lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0$

Hence, $\{g_n(x)\}$ converges uniformly on $\mathbb{R}$.
Hence, option (A) is incorrect.

$g_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$

$$\lim_{n\to\infty} g_n'(x) = h(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Hence, $\{g_n'\}$ does not converge pointwise to a continuous function on $\mathbb{R}$.

But $\{g_n'\}$ will converge uniformly on any closed interval that does not contain 0 because the limit function is not continuous at 0 only.
Hence, options (B) and (C) are incorrect.
Hence, option (D) is correct.

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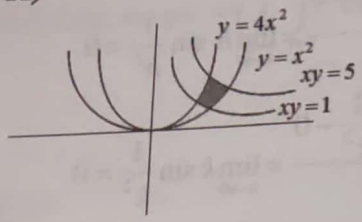
124. Solution: (Ans: 12)



Let $u = \frac{y}{x^2}$, $v = xy$ to transform the given integral in terms of u and v and limit changes to

$1 \le u \le 4;\ 1 \le v \le 5$

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$\text{Now } \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-2y}{x^3} & \frac{1}{x^2} \\ y & x \end{vmatrix} = \frac{-3y}{x^2}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = -\frac{x^2}{3y}$$

$$\therefore \iint_R \frac{y^2}{x} dx\,dy = \frac{1}{3}\int_{v=1}^{5} \int_{u=1}^{4} v\,du\,dv = \frac{1}{3}\int_1^5 v dv \int_1^4 du = 12$$

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